NOTE: Beviser i differentialregningen

KBJ, maj 2024 *2u MA*

Sætning 1: $f(x) = a \cdot x + b \Longrightarrow f'(x_0) = a$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x)}{\Delta x} = \frac{(a(x_0 + \Delta x) + b) - (ax_0 + b)}{\Delta x}$$

$$\frac{(a(x_0 + \Delta x) + b) - (ax_0 + b)}{\Delta x} = \frac{ax_0 + a\Delta x + b - ax_0 - b}{\Delta x} = \frac{a\Delta x}{\Delta x} = a$$

$$\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}(a) = a = f'(x_0)$$ *QED*

Sætning 2: $f(x) = x^2 \Longrightarrow f'(x_0) = 2x_0$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{(x_0 + \Delta x)^2 - x_0^2}{\Delta x}$$

$$\frac{(x_0 + \Delta x)^2 - x_0^2}{\Delta x} = \frac{x_0^2 + 2x_0\Delta x + \Delta x^2 - x_0^2}{\Delta x} = \frac{2x_0\Delta x + \Delta x^2}{\Delta x} = \frac{2x_0\Delta x}{\Delta x} + \frac{\Delta x^2}{\Delta x} = 2x_0 + \Delta x$$

$$\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}(2x_0 + \Delta x) = 2x_0 + 0 = 2x_0 = f'(x_0)$$ *QED*

Sætning 3: $f(x) = x^3 \Longrightarrow f'(x_0) = 3x_0^2$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{(x_0 + \Delta x)^3 - x_0^3}{\Delta x}$$

$$\frac{(x_0 + \Delta x)^3 - x_0^3}{\Delta x} = \frac{x_0^3 + 3x_0^2\Delta x + 3x_0\Delta x^2 + \Delta x^3 - x_0^3}{\Delta x} = \frac{3x_0^2\Delta x + 3x_0\Delta x^2 + \Delta x^3}{\Delta x}$$
$$= 3x_0^2 + 3x_0\Delta x + \Delta x^2$$

$$\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}(3x_0^2 + 3x_0\Delta x + \Delta x^2) = 3x_0^2 + 3x_0 \cdot 0 + 0 = 3x_0^2 = f'(x_0)$$ *QED*

Sætning 4: $f(x) = x^4 \Longrightarrow f'(x_0) = 4x_0^3$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{(x_0 + \Delta x)^4 - x_0^4}{\Delta x}$$

$$\frac{(x_0 + \Delta x)^4 - x_0^4}{\Delta x} = \frac{x_0^4 + 4x_0^3\Delta x + 6x_0^2\Delta x^2 + 4x_0\Delta x^3 + \Delta x^4 - x_0^4}{\Delta x} =$$
$$\frac{4x_0^3\Delta x + 6x_0^2\Delta x^2 + 4x_0\Delta x^3 + \Delta x^4}{\Delta x} = 4x_0^3 + 6x_0^2\Delta x + 4x_0\Delta x^2 + \Delta x^3$$

$$\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}(4x_0^3 + 6x_0^2\Delta x + 4x_0\Delta x^2 + \Delta x_0^3) = 4x_0^3 + 6x_0^2 \cdot 0 + 4x_0 \cdot 0 + 0 = 4x_0^3 = f'(x_0)$$
QED

Sætning 5: $f(x) = \frac{1}{x} \Longrightarrow f'(x_0) = -\frac{1}{x_0^2},\ x_0 \neq 0$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{\dfrac{1}{x_0 + \Delta x} - \dfrac{1}{x_0}}{\Delta x}$$

$$\frac{\frac{1}{x_0 + \Delta x} - \frac{1}{x_0}}{\Delta x} = \frac{\frac{x_0}{x_0(x_0 + \Delta x)} - \frac{x_0 + \Delta x}{x_0(x_0 + \Delta x)}}{\Delta x} = \frac{\frac{x_0 - x_0 - \Delta x}{x_0(x_0 + \Delta x)}}{\Delta x} = \frac{\frac{-\Delta x}{x_0(x_0 + \Delta x)}}{\Delta x} = \frac{1}{\Delta x} \cdot \frac{-\Delta x}{x_0(x_0 + \Delta x)} = \frac{-1}{x_0(x_0 + \Delta x)}$$

$$\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left(\frac{-1}{x_0(x_0 + \Delta x)}\right) = \frac{-1}{x_0(x_0 + 0)} = \frac{-1}{x_0 \cdot x_0} = -\frac{1}{x_0^2} = f'(x_0),\ \ x \neq 0$$ *QED*

Sætning 6: $f(x) = \sqrt{x} \Longrightarrow f'(x_0) = \frac{1}{2\sqrt{x_0}},\ \ x > 0$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{\sqrt{x_0 + \Delta x} - \sqrt{x_0}}{\Delta x}$$

$$\frac{\sqrt{x_0 + \Delta x} - \sqrt{x_0}}{\Delta x} = \frac{\left(\sqrt{x_0 + \Delta x} - \sqrt{x}_0\right)\left(\sqrt{x_0 + \Delta x} + \sqrt{x_0}\right)}{\Delta x \cdot \left(\sqrt{x_0 + \Delta x} + \sqrt{x_0}\right)} = \frac{\sqrt{x_0 + \Delta x}^2 - \sqrt{x}_0{}^2}{\Delta x \cdot \left(\sqrt{x_0 + \Delta x} + \sqrt{x_0}\right)} =$$
$$\frac{x_0 + \Delta x - x_0}{\Delta x \cdot \left(\sqrt{x_0 + \Delta x} + \sqrt{x_0}\right)} = \frac{\Delta x}{\Delta x \cdot \left(\sqrt{x_0 + \Delta x} + \sqrt{x_0}\right)} = \frac{1}{\left(\sqrt{x_0 + \Delta x} + \sqrt{x}_0\right)}$$

$$\lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left(\frac{1}{\left(\sqrt{x_0 + \Delta x} + \sqrt{x_0}\right)}\right) = \frac{1}{\left(\sqrt{x_0 + 0} + \sqrt{x_0}\right)} = \frac{1}{\left(\sqrt{x_0} + \sqrt{x_0}\right)} = \frac{1}{2\sqrt{x_0}} = f'(x_0),\ \ x_0 > 0$$ *QED*

For $x_0 = 0$ fås: $\frac{1}{\left(\sqrt{0 + \Delta x} + \sqrt{0}\right)} = \frac{1}{\sqrt{\Delta x}} \to \infty$ for $\Delta x \to 0$. Der findes altså ikke en grænseværdi for $x_0 = 0$.

Sætning 7: For g differentiabel i x_0 gælder: $f(x) = k \cdot g(x) \Longrightarrow f'(x_0) = k \cdot g'(x_0)$.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{k \cdot g(x_0 + \Delta x) - k \cdot g(x_0)}{\Delta x} = \frac{k \cdot \big(g(x_0 + \Delta x) - g(x_0)\big)}{\Delta x} =$$
$$k \cdot \frac{g(x_0 + \Delta x) - g(x_0)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0} \left(k \cdot \frac{g(x_0+\Delta x) - g(x)}{\Delta x}\right) = k \cdot \lim_{\Delta x \to 0} \left(\frac{g(x_0+\Delta x) - g(x_0)}{\Delta x}\right) = k \cdot g'(x_0) = f'(x_0) \qquad QED$$

Sætning 8: For g og h differentiable gælder: $f(x_0) = g(x_0) + h(x_0) \Longrightarrow f'(x_0) = g'(x_0) + h'(x_0)$.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{\big(g(x_0 + \Delta x) + h(x_0 + \Delta x)\big) - \big(g(x_0) + h(x_0)\big)}{\Delta x} =$$
$$\frac{g(x_0 + \Delta x) + h(x_0 + \Delta x) - g(x_0) - h(x_0)}{\Delta x} = \frac{g(x_0 + \Delta x) - g(x_0) + h(x_0 + \Delta x) - h(x_0)}{\Delta x} =$$
$$\frac{g(x_0 + \Delta x) - g(x_0)}{\Delta x} + \frac{h(x_0 + \Delta x) - h(x_0)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0} \left(\frac{g(x_0+\Delta x) - g(x_0)}{\Delta x} + \frac{h(x_0+\Delta x) - h(x_0)}{\Delta x}\right) = \lim_{\Delta x \to 0} \left(\frac{g(x_0+\Delta x) - g(x_0)}{\Delta x}\right) + \lim_{\Delta x \to 0} \left(\frac{h(x_0+\Delta x) - h(x_0)}{\Delta x}\right) =$$
$$g'(x_0) + h'(x_0) = f'(x_0) \qquad QED$$

Sætning 9: For g og h differentiable gælder: $f(x) = g(x) - h(x) \Longrightarrow f'(x_0) = g'(x_0) - h'(x_0)$.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{\big(g(x_0 + \Delta x) - h(x_0 + \Delta x)\big) - \big(g(x_0) - h(x_0)\big)}{\Delta x} =$$
$$\frac{g(x_0 + \Delta x) - h(x_0 + \Delta x) - g(x_0) + h(x_0)}{\Delta x} = \frac{g(x_0 + \Delta x) - g(x_0) - h(x_0 + \Delta x) + h(x_0)}{\Delta x} =$$
$$\frac{g(x_0 + \Delta x) - g(x_0)}{\Delta x} + \frac{-h(x_0 + \Delta x) + h(x_0)}{\Delta x} = \frac{g(x_0 + \Delta x) - g(x_0)}{\Delta x} - \frac{h(x_0 + \Delta x) - h(x_0)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0} \left(\frac{g(x_0+\Delta x) - g(x_0)}{\Delta x} - \frac{h(x_0+\Delta x) - h(x_0)}{\Delta x}\right) = \lim_{\Delta x \to 0} \left(\frac{g(x_0+\Delta x) - g(x_0)}{\Delta x}\right) - \lim_{\Delta x \to 0} \left(\frac{h(x_0+\Delta x) - h(x_0)}{\Delta x}\right) =$$
$$g'(x_0) - h'(x_0) = f'(x_0) \qquad QED$$

Sætning 10: For g og h differentiable gælder:
$f(x) = g(x) \cdot h(x) \Rightarrow f'(x_0) = g'(x_0) \cdot h(x_0) + g(x_0) \cdot h'(x_0)$.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{g(x_0 + \Delta x) \cdot h(x_0 + \Delta x) - g(x_0) \cdot h(x_0)}{\Delta x} =$$

$$\frac{g(x_0 + \Delta x) \cdot h(x_0 + \Delta x) - g(x_0) \cdot h(x_0) + g(x_0 + \Delta x) \cdot h(x_0) - g(x_0 + \Delta x) \cdot h(x_0)}{\Delta x} =$$

$$\frac{g(x_0 + \Delta x) \cdot h(x_0) - g(x_0) \cdot h(x_0) + g(x_0 + \Delta x) \cdot h(x_0 + \Delta x) - g(x_0 + \Delta x) \cdot h(x_0)}{\Delta x} =$$

$$\frac{\big(g(x_0 + \Delta x) - g(x_0)\big) \cdot h(x_0) + g(x_0 + \Delta x) \cdot \big(h(x_0 + \Delta x) - h(x_0)\big)}{\Delta x} =$$

$$\frac{\big(g(x_0 + \Delta x) - g(x_0)\big) \cdot h(x_0)}{\Delta x} + \frac{g(x_0 + \Delta x) \cdot \big(h(x_0 + \Delta x) - h(x_0)\big)}{\Delta x} =$$

$$\frac{g(x_0 + \Delta x) - g(x_0)}{\Delta x} \cdot h(x_0) + g(x_0 + \Delta x) \cdot \frac{h(x_0 + \Delta x) - h(x_0)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0} \left(\frac{g(x_0 + \Delta x) - g(x_0)}{\Delta x} \cdot h(x_0) + g(x_0 + \Delta x) \cdot \frac{h(x_0 + \Delta x) - h(x_0)}{\Delta x}\right) =$$

$$\lim_{\Delta x \to 0} \left(\frac{g(x_0 + \Delta x) - g(x_0)}{\Delta x} \cdot h(x_0)\right) + \lim_{\Delta x \to 0} \left(g(x_0 + \Delta x) \cdot \frac{h(x_0 + \Delta x) - h(x_0)}{\Delta x}\right) =$$

$$\lim_{\Delta x \to 0} \left(\frac{g(x_0 + \Delta x) - g(x_0)}{\Delta x}\right) \cdot \lim_{\Delta x \to 0} \big(h(x_0)\big) + \lim_{\Delta x \to 0} \big(g(x_0 + \Delta x)\big) \cdot \lim_{\Delta x \to 0} \left(\frac{h(x_0 + \Delta x) - h(x_0)}{\Delta x}\right) =$$

$$g'(x_0) \cdot h(x_0) + g(x_0) \cdot h'(x_0) = f'(x_0) \qquad QED$$

Sætning 11: For $f(x) = x^n,\ n \in \mathbb{N}$, gælder $f'(x) = n \cdot x^{n-1}$

Antaget at *hvis* $f(x) = x^n$, *så* $f'(x) = n \cdot x^{n-1}$, da fås for $f(x) = x^{n+1} = x \cdot x^n$:

$(x \cdot x^n)' = (x)' \cdot x^n + x \cdot (x^n)' = 1 \cdot x^n + x \cdot n \cdot x^{n-1} = 1 \cdot x^n + n \cdot x^n = (1+n) \cdot x^n$

Hvis sætning 11 gælder for n, gælder den altså også for $n+1$.

Af sætning 1 ses at $(x^1)' = 1 = 1 \cdot x^0$. Altså gælder sætning 11 for $n = 1$.

Ved induktion er det således bevist, at sætning 11 gælder for alle $n \in \mathbb{N}$.

QED

Sætning 12: For $f(x) = x^a$, $a \in \mathbb{R}$, gælder $f'(x) = a \cdot x^{a-1}$.

Det udnyttes at $x = \mathrm{e}^{\ln(x)}$.

$f(x) = x^a = \left(\mathrm{e}^{\ln(x)}\right)^a = \mathrm{e}^{a \cdot \ln(x)}$

$f'(x) = (a \cdot \ln(x))' \cdot \mathrm{e}^{a \cdot \ln(x)} = a \cdot \frac{1}{x} \cdot e^{a \cdot \ln(x)} = a \cdot \frac{1}{x} \cdot e^{\ln(x^a)} = a \cdot \frac{1}{x} \cdot x^a = a \cdot x^{a-1}$

QED