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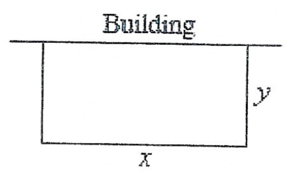
Section 4-8 : Optimization

Example 1 We need to enclose a rectangular field with a fence. We have 500 feet of fencing material and a building is on one side of the field and so won't need any fencing. Determine the dimensions of the field that will enclose the largest area.

Solution

In all of these problems we will have two functions. The first is the function that we are actually trying to optimize and the second will be the constraint. Sketching the situation will often help us to arrive at these equations so let's do that.



In this problem we want to maximize the area of a field and we know that will use 500 ft of fencing material. So, the area will be the function we are trying to optimize and the amount of fencing is the constraint. The two equations for these are,

$$\text{Maximize} : A = xy$$
$$\text{Constraint} : 500 = x + 2y$$

Okay, we know how to find the largest or smallest value of a function provided it's only got a single variable. The area function (as well as the constraint) has two variables in it and so what we know about finding absolute extrema won't work. However, if we solve the constraint for one of the two variables we can substitute this into the area and we will then have a function of a single variable.

So, let's solve the constraint for *x*. Note that we could have just as easily solved for *y* but that would have led to fractions and so, in this case, solving for *x* will probably be best.

$$x = 500 - 2y$$

Substituting this into the area function gives a function of *y*.

$$A(y) = (500 - 2y)y = 500y - 2y^2$$

Now we want to find the largest value this will have on the interval $[0, 250]$. The limits in this interval corresponds to taking $y = 0$ (*i.e.* no sides to the fence) and $y = 250$ (*i.e.* only two sides and no width, also if there are two sides each must be 250 ft to use the whole 500ft).

Note that the endpoints of the interval won't make any sense from a physical standpoint if we actually want to enclose some area because they would both give zero area. They do, however, give us a set of limits on *y* and so the Extreme Value Theorem tells us that we will have a maximum value of the area somewhere between the two endpoints. Having these limits will also mean that we can use the process we discussed in the Finding Absolute Extrema section earlier in the chapter to find the maximum value of the area.

So, recall that the maximum value of a continuous function (which we've got here) on a closed interval (which we also have here) will occur at critical points and/or end points. As we've already pointed out the end points in this case will give zero area and so don't make any sense. That means our only option will be the critical points.

So, let's get the derivative and find the critical points.

$$A'(y) = 500 - 4y$$

Setting this equal to zero and solving gives a lone critical point of $y = 125$. Plugging this into the area gives an area of $A(125) = 31250\,\text{ft}^2$. So according to the method from Absolute Extrema section this must be the largest possible area, since the area at either endpoint is zero.

Finally, let's not forget to get the value of *x* and then we'll have the dimensions since this is what the problem statement asked for. We can get the *x* by plugging in our *y* into the constraint.

$$x = 500 - 2(125) = 250$$

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Example 3 We want to construct a box with a square base and we only have 10 m^2 of material to use in construction of the box. Assuming that all the material is used in the construction process determine the maximum volume that the box can have.

Solution

This example is in many ways the exact opposite of the previous example. In this case we want to optimize the volume and the constraint this time is the amount of material used. We don't have a cost here, but if you think about it the cost is nothing more than the amount of material used times a cost and so the amount of material and cost are pretty much tied together. If you can do one you can do the other as well. Note as well that the amount of material used is really just the surface area of the box.

As always, let's start off with a quick sketch of the box.

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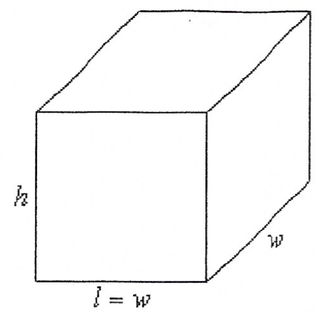

Now, as mentioned above we want to maximize the volume and the amount of material is the constraint so here are the equations we'll need.

$$\text{Maximize} : V = lwh = w^2h$$

$$\text{Constraint} : 10 = 2lw + 2wh + 2lh = 2w^2 + 4wh$$

We'll solve the constraint for *h* and plug this into the equation for the volume.

$$h = \frac{10-2w^2}{4w} = \frac{5-w^2}{2w} \qquad \Rightarrow \qquad V(w) = w^2\left(\frac{5-w^2}{2w}\right) = \frac{1}{2}\left(5w - w^3\right)$$

Here are the first and second derivatives of the volume function.

$$V'(w) = \tfrac{1}{2}\left(5-3w^2\right) \qquad\qquad V''(w) = -3w$$

Note as well here that provided $w > 0$, which from a physical standpoint we know must be true for the width of the box, then the volume function will be concave down and so if we get a single critical point then we know that it will have to be the value that gives the absolute maximum.

Setting the first derivative equal to zero and solving gives us the two critical points,

$$w = \pm\sqrt{\frac{5}{3}} = \pm 1.2910$$

In this case we can exclude the negative critical point since we are dealing with a length of a box and we know that these must be positive. Do not however get into the habit of just excluding any negative critical point. There are problems where negative critical points are perfectly valid possible solutions.

Now, as noted above we got a single critical point, 1.2910, and so this must be the value that gives the maximum volume and since the maximum volume is all that was asked for in the problem statement the answer is then : $V(1.2910) = 2.1517\,\text{m}^3$.

Note that we could also have noted here that if $0 < w < 1.2910$ then $V'(w) > 0$ (using a test point we have $V'(1) = 1 > 0$) and likewise if $w > 1.2910$ then $V'(w) < 0$ (using a test point we have $V'(2) = -\frac{7}{2} < 0$) and so if we are to the left of the critical point the volume is always increasing and if we are to the right of the critical point the volume is always decreasing and so by the Method 2 above we can also see that the single critical point must give the absolute maximum of the volume.

Finally, even though these weren't asked for here are the dimension of the box that gives the maximum volume.

$$l = w = 1.2910 \qquad h = \frac{5 - 1.2910^2}{2(1.2910)} = 1.2910$$

So, it looks like in this case we actually have a perfect cube.