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Calculus I

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Definitions

Given a function, $f(x)$, an **anti-derivative** of $f(x)$ is any function $F(x)$ such that

$$F'(x) = f(x)$$

If $F(x)$ is any anti-derivative of $f(x)$ then the most general anti-derivative of $f(x)$ is called an **indefinite integral** and denoted,

$$\int f(x)\,dx = F(x) + c, \quad c \text{ is any constant}$$

In this definition the $\int$ is called the **integral symbol**, $f(x)$ is called the **integrand**, *x* is called the **integration variable** and the "*c*" is called the **constant of integration**.

Example 3 Integrate $\int \sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)dt$.

Solution

There are several ways to do this integral and most of them require the next section. However, there is a way to do this integral using only the material from this section. All that is required is to remember the trig formula that we can use to simplify the integrand up a little. Recall the following double angle formula.

$$\sin(2t) = 2\sin t\cos t$$

A small rewrite of this formula gives,

$$\sin t\cos t = \tfrac{1}{2}\sin(2t)$$

If we now replace all the *t*'s with $\frac{t}{2}$ we get,

$$\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right) = \frac{1}{2}\sin(t)$$

Using this formula, the integral becomes something we can do.

$$\begin{aligned}\int \sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)dt &= \int \frac{1}{2}\sin(t)\,dt \\ &= -\frac{1}{2}\cos(t)+c\end{aligned}$$

As noted earlier there is another method for doing this integral. In fact, there are two alternate methods. To see all three check out the section on **Constant of Integration** in the Extras chapter but be aware that the other two do require the material covered in the next section.

The formula/simplification in the previous problem is a nice "trick" to remember. It can be used on occasion to greatly simplify some problems.

There is one more set of examples that we should do before moving out of this section.

Example 4 Given the following information determine the function $f(x)$.

(a) $f'(x) = 4x^3 - 9 + 2\sin x + 7\mathbf{e}^x,\ f(0) = 15$

(b) $f''(x) = 15\sqrt{x} + 5x^3 + 6,\ \ f(1) = -\frac{5}{4},\ \ f(4) = 404$

Solution

In both of these we will need to remember that

$$f(x) = \int f'(x)\,dx$$

Also note that because we are giving values of the function at specific points we are also going to be determining what the constant of integration will be in these problems.

(a) $f'(x) = 4x^3 - 9 + 2\sin x + 7\mathbf{e}^x$, $f(0) = 15$

The first step here is integrating to determine the most general possible $f(x)$.

$$\begin{aligned} f(x) &= \int 4x^3 - 9 + 2\sin x + 7\mathbf{e}^x \, dx \\ &= x^4 - 9x - 2\cos x + 7\mathbf{e}^x + c \end{aligned}$$

Now we have a value of the function so let's plug in $x = 0$ and determine the value of the constant of integration *c*.

$$\begin{aligned} 15 = f(0) &= 0^4 - 9(0) - 2\cos(0) + 7\mathbf{e}^0 + c \\ &= -2 + 7 + c \\ &= 5 + c \end{aligned}$$

So, from this it looks like $c = 10$. This means that the function is,

$$f(x) = x^4 - 9x - 2\cos x + 7\mathbf{e}^x + 10$$

(b) $f''(x) = 15\sqrt{x} + 5x^3 + 6$, $f(1) = -\frac{5}{4}$, $f(4) = 404$

This one is a little different form the first one. In order to get the function we will need the first derivative and we have the second derivative. We can however, use an integral to get the first derivative from the second derivative, just as we used an integral to get the function from the first derivative.

So, let's first get the most general possible first derivative by integrating the second derivative.

$$\begin{aligned} f'(x) &= \int f''(x)\,dx \\ &= \int 15x^{\frac{1}{2}} + 5x^3 + 6\,dx \\ &= 15\left(\frac{2}{3}\right)x^{\frac{3}{2}} + \frac{5}{4}x^4 + 6x + c \\ &= 10x^{\frac{3}{2}} + \frac{5}{4}x^4 + 6x + c \end{aligned}$$

Don't forget the constant of integration!

We can now find the most general possible function by integrating the first derivative which we found above.

$$\begin{aligned} f(x) &= \int 10x^{\frac{3}{2}} + \frac{5}{4}x^4 + 6x + c\,dx \\ &= 4x^{\frac{5}{2}} + \frac{1}{4}x^5 + 3x^2 + cx + d \end{aligned}$$

Do not get excited about integrating the *c*. It's just a constant and we know how to integrate constants. Also, there will be no reason to think the constants of integration from the integration in each step will be the same and so we'll need to call each constant of integration something different, *d* in this case.

Now, plug in the two values of the function that we've got.

$$-\frac{5}{4}=f(1)=4+\frac{1}{4}+3+c+d=\frac{29}{4}+c+d$$

$$404=f(4)=4(32)+\frac{1}{4}(1024)+3(16)+c(4)+d=432+4c+d$$

This gives us a system of two equations in two unknowns that we can solve.

$$\begin{aligned}-\frac{5}{4}&=\frac{29}{4}+c+d\\ 404&=432+4c+d\end{aligned} \qquad \Rightarrow \qquad \begin{aligned}c&=-\frac{13}{2}\\ d&=-2\end{aligned}$$

The function is then,

$$f(x)=4x^{\frac{5}{2}}+\frac{1}{4}x^{5}+3x^{2}-\frac{13}{2}x-2$$

Substitution Rule

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du, \qquad \text{where, } u = g(x)$$

A natural question at this stage is how to identify the correct substitution. Unfortunately, the answer is it depends on the integral. However, there is a general rule of thumb that will work for many of the integrals that we're going to be running across.

When faced with an integral we'll ask ourselves what we know how to integrate. With the integral above we can quickly recognize that we know how to integrate

$$\int \sqrt[4]{x}\,dx$$

However, we didn't have just the root we also had stuff in front of the root and (more importantly in this case) stuff under the root. Since we can only integrate roots if there is just an *x* under the root a good first guess for the substitution is then to make *u* be the stuff under the root.

Another way to think of this is to ask yourself if you were to differentiate the integrand (we're not of course, but just for a second pretend that we were) is there a chain rule and what is the inside function for the chain rule. If there is a chain rule (for a derivative) then there is a pretty good chance that the inside function will be the substitution that will allow us to do the integral.

We will have to be careful however. There are times when using this general rule can get us in trouble or overly complicate the problem. We'll eventually see problems where there are more than one "inside function" and/or integrals that will look very similar and yet use completely different substitutions. The reality is that the only way to really learn how to do substitutions is to just work lots of problems and eventually you'll start to get a feel for how these work and you'll find it easier and easier to identify the proper substitutions.

Now, with that out of the way we should ask the following question. How, do we know if we got the correct substitution? Well, upon computing the differential and actually performing the substitution every *x* in the integral (including the *x* in the dx) must disappear in the substitution process and the only letters left should be *u*'s (including a du) and we should be left with an integral that we can do.

If there are *x*'s left over or we have an integral that cannot be evaluated then there is a pretty good chance that we chose the wrong substitution. Unfortunately, however there is at least one case (we'll be seeing an example of this in the next section) where the correct substitution will actually leave some *x*'s and we'll need to do a little more work to get it to work.

Again, it cannot be stressed enough at this point that the only way to really learn how to do substitutions is to just work lots of problems. There are lots of different kinds of problems and after working many problems you'll start to get a real feel for these problems and after you work enough of them you'll reach the point where you'll be able to do simple substitutions in your head without having to actually write anything down.

As a final note we should point out that often (in fact in almost every case) the differential will not appear exactly in the integrand as it did in the example above and sometimes we'll need to do some manipulation of the integrand and/or the differential to get all the *x*'s to disappear in the substitution.

Let's work some examples so we can get a better idea on how the substitution rule works.

Example 1 Evaluate each of the following integrals.

(a) $\displaystyle\int\left(1-\frac{1}{w}\right)\cos\left(w-\ln w\right)dw$

(b) $\displaystyle\int 3\left(8y-1\right)\mathbf{e}^{4y^2-y}\,dy$

(c) $\displaystyle\int x^2\left(3-10x^3\right)^4\,dx$

(d) $\displaystyle\int\frac{x}{\sqrt{1-4x^2}}dx$

Solution

(a) $\displaystyle\int\left(1-\frac{1}{w}\right)\cos\left(w-\ln w\right)dw$

In this case it looks like we have a cosine with an inside function and so let's use that as the substitution.

$$u = w-\ln w \qquad\qquad du=\left(1-\frac{1}{w}\right)dw$$

So, as with the first example we worked the stuff in front of the cosine appears exactly in the differential. The integral is then,

$$\begin{aligned}\int\left(1-\frac{1}{w}\right)\cos\left(w-\ln w\right)dw &= \int\cos\left(u\right)du\\ &=\sin\left(u\right)+c\\ &=\sin\left(w-\ln w\right)+c\end{aligned}$$

Don't forget to go back to the original variable in the problem.

(b) $\displaystyle\int 3\left(8y-1\right)\mathbf{e}^{4y^2-y}\,dy$

Again, it looks like we have an exponential function with an inside function (*i.e.* the exponent) and it looks like the substitution should be,

$$u=4y^2-y \qquad\qquad du=\left(8y-1\right)dy$$

Now, with the exception of the 3 the stuff in front of the exponential appears exactly in the differential. Recall however that we can factor the 3 out of the integral and so it won't cause any problems. The integral is then,

$$\begin{aligned}\int 3\left(8y-1\right)\mathbf{e}^{4y^2-y}\,dy &= 3\int\mathbf{e}^u\,du\\ &=3\mathbf{e}^u+c\\ &=3\mathbf{e}^{4y^2-y}+c\end{aligned}$$

(c) $\displaystyle\int x^2\left(3-10x^3\right)^4\,dx$

In this case it looks like the following should be the substitution.

$$u = 3 - 10x^3 \qquad\qquad du = -30x^2\,dx$$

Okay, now we have a small problem. We've got an x^2 out in front of the parenthesis but we don't have a "-30". This is not really the problem it might appear to be at first. We will simply rewrite the differential as follows.

$$x^2\,dx = -\frac{1}{30}du$$

With this we can now substitute the $x^2\,dx$ away. In the process we will pick up a constant, but that isn't a problem since it can always be factored out of the integral.

We can now do the integral.

$$\begin{aligned}\int x^2\left(3-10x^3\right)^4 dx &= \int\left(3-10x^3\right)^4 x^2\,dx\\ &= \int u^4\left(-\frac{1}{30}\right)du\\ &= -\frac{1}{30}\left(\frac{1}{5}\right)u^5 + c\\ &= -\frac{1}{150}\left(3-10x^3\right)^5 + c\end{aligned}$$

Note that in most problems when we pick up a constant as we did in this example we will generally factor it out of the integral in the same step that we substitute it in.

(d) $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}dx$

In this example don't forget to bring the root up to the numerator and change it into fractional exponent form. Upon doing this we can see that the substitution is,

$$u = 1-4x^2 \qquad\qquad du = -8x\,dx \qquad \Rightarrow \qquad x\,dx = -\frac{1}{8}du$$

The integral is then,

$$\begin{aligned}\int \frac{x}{\sqrt{1-4x^2}}dx &= \int x\left(1-4x^2\right)^{-\frac{1}{2}}dx\\ &= -\frac{1}{8}\int u^{-\frac{1}{2}}du\\ &= -\frac{1}{4}u^{\frac{1}{2}} + c\\ &= -\frac{1}{4}\left(1-4x^2\right)^{\frac{1}{2}} + c\end{aligned}$$

In the previous set of examples the substitution was generally pretty clear. There was exactly one term that had an "inside function" and so there wasn't really much in the way of options for the substitution.

Let's take a look at some more complicated problems to make sure we don't come to expect all substitutions are like those in the previous set of examples.

Example 2 Evaluate each of the following integrals.

(a) $\int \sin(1-x)\left(2-\cos(1-x)\right)^4 dx$

(b) $\int \cos(3z)\sin^{10}(3z)\,dz$

Solution

(a) $\int \sin(1-x)\left(2-\cos(1-x)\right)^4 dx$

In this problem there are two "inside functions". There is the $1-x$ that is inside the two trig functions and there is also the term that is raised to the 4th power.

There are two ways to proceed with this problem. The first idea that many students have is substitute the $1-x$ away. There is nothing wrong with doing this but it doesn't eliminate the problem of the term to the 4th power. That's still there and if we used this idea we would then need to do a second substitution to deal with that.

The second (and much easier) way of doing this problem is to just deal with the stuff raised to the 4th power and see what we get. The substitution in this case would be,

$$u = 2-\cos(1-x) \qquad du = -\sin(1-x)\,dx \qquad \Rightarrow \qquad \sin(1-x)\,dx = -du$$

Two things to note here. First, don't forget to correctly deal with the "-". A common mistake at this point is to lose it. Secondly, notice that the $1-x$ turns out to not really be a problem after all. Because the $1-x$ was "buried" in the substitution that we actually used it was also taken care of at the same time. The integral is then,

$$\begin{aligned}\int \sin(1-x)\left(2-\cos(1-x)\right)^4 dx &= -\int u^4\,du \\ &= -\frac{1}{5}u^5 + c \\ &= -\frac{1}{5}\left(2-\cos(1-x)\right)^5 + c\end{aligned}$$

As seen in this example sometimes there will seem to be two substitutions that will need to be done however, if one of them is buried inside of another substitution then we'll only really need to do one. Recognizing this can save a lot of time in working some of these problems.

(b) $\int \cos(3z)\sin^{10}(3z)\,dz$

This one is a little tricky at first. We can see the correct substitution by recalling that,

$$\sin^{10}(3z) = \left(\sin(3z)\right)^{10}$$

Using this it looks like the correct substitution is,

$$u = \sin(3z) \qquad du = 3\cos(3z)dz \qquad \Rightarrow \qquad \cos(3z)dz = \frac{1}{3}du$$

Notice that we again had two apparent substitutions in this integral but again the $3z$ is buried in the substitution we're using and so we didn't need to worry about it.
Here is the integral.

$$\begin{aligned}\int \cos(3z)\sin^{10}(3z)dz &= \frac{1}{3}\int u^{10}\,du \\ &= \frac{1}{3}\left(\frac{1}{11}\right)u^{11} + c \\ &= \frac{1}{33}\sin^{11}(3z) + c\end{aligned}$$

Note that the one third in front of the integral came about from the substitution on the differential and we just factored it out to the front of the integral. This is what we will usually do with these constants.

Section 5-8 : Substitution Rule for Definite Integrals

We now need to go back and revisit the substitution rule as it applies to definite integrals. At some level there really isn't a lot to do in this section. Recall that the first step in doing a definite integral is to compute the indefinite integral and that hasn't changed. We will still compute the indefinite integral first. This means that we already know how to do these. We use the substitution rule to find the indefinite integral and then do the evaluation.

There are however, two ways to deal with the evaluation step. One of the ways of doing the evaluation is the probably the most obvious at this point, but also has a point in the process where we can get in trouble if we aren't paying attention.

Let's work an example illustrating both ways of doing the evaluation step.

Example 1 Evaluate the following definite integral.

$$\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt$$

Solution

Let's start off looking at the first way of dealing with the evaluation step. We'll need to be careful with this method as there is a point in the process where if we aren't paying attention we'll get the wrong answer.

Solution 1 :

We'll first need to compute the indefinite integral using the substitution rule. Note however, that we will constantly remind ourselves that this is a definite integral by putting the limits on the integral at each step. Without the limits it's easy to forget that we had a definite integral when we've gotten the indefinite integral computed.

In this case the substitution is,

$$u = 1-4t^3 \qquad du = -12t^2dt \qquad \Rightarrow \qquad t^2dt = -\frac{1}{12}du$$

Plugging this into the integral gives,

$$\begin{aligned}\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt &= -\frac{1}{6}\int_{-2}^{0} u^{\frac{1}{2}}\,du \\ &= -\frac{1}{9}u^{\frac{3}{2}}\Big|_{-2}^{0}\end{aligned}$$

Notice that we didn't do the evaluation yet. This is where the potential problem arises with this solution method. The limits given here are from the original integral and hence are values of *t*. We have *u*'s in our solution. We can't plug values of *t* in for *u*.

Therefore, we will have to go back to *t*'s before we do the substitution. This is the standard step in the substitution process, but it is often forgotten when doing definite integrals. Note as well that in this case, if we don't go back to t's we will have a small problem in that one of the evaluations will end up giving us a complex number.

So, finishing this problem gives,

$$\begin{aligned}\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt &= -\frac{1}{9}\left(1-4t^3\right)^{\frac{3}{2}}\Big|_{-2}^{0}\\ &= -\frac{1}{9}-\left(-\frac{1}{9}(33)^{\frac{3}{2}}\right)\\ &= \frac{1}{9}\left(33\sqrt{33}-1\right)\end{aligned}$$

So, that was the first solution method. Let's take a look at the second method.

Solution 2 :
Note that this solution method isn't really all that different from the first method. In this method we are going to remember that when doing a substitution we want to eliminate all the t's in the integral and write everything in terms of u.

When we say all here we really mean all. In other words, remember that the limits on the integral are also values of t and we're going to convert the limits into u values. Converting the limits is pretty simple since our substitution will tell us how to relate t and u so all we need to do is plug in the original t limits into the substitution and we'll get the new u limits.

Here is the substitution (it's the same as the first method) as well as the limit conversions.

$$u = 1-4t^3 \qquad du = -12t^2dt \qquad \Rightarrow \qquad t^2dt = -\frac{1}{12}du$$

$$t = -2 \qquad \Rightarrow \qquad u = 1-4(-2)^3 = 33$$

$$t = 0 \qquad \Rightarrow \qquad u = 1-4(0)^3 = 1$$

The integral is now,

$$\begin{aligned}\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt &= -\frac{1}{6}\int_{33}^{1} u^{\frac{1}{2}}\,du\\ &= -\frac{1}{9}u^{\frac{3}{2}}\Big|_{33}^{1}\end{aligned}$$

As with the first method let's pause here a moment to remind us what we're doing. In this case, we've converted the limits to u's and we've also got our integral in terms of u's and so here we can just plug the limits directly into our integral. Note that in this case we won't plug our substitution back in. Doing this here would cause problems as we would have t's in the integral and our limits would be u's. Here's the rest of this problem.

$$\int_{-2}^{0} 2t^2\sqrt{1-4t^3}\,dt = -\frac{1}{9}u^{\frac{3}{2}}\Big|_{33}^{1}$$

$$= -\frac{1}{9} - \left(-\frac{1}{9}(33)^{\frac{3}{2}}\right) = \frac{1}{9}\left(33\sqrt{33}-1\right)$$

We got exactly the same answer and this time didn't have to worry about going back to *t*'s in our answer.

So, we've seen two solution techniques for computing definite integrals that require the substitution rule. Both are valid solution methods and each have their uses. We will be using the second almost exclusively however since it makes the evaluation step a little easier.