CHAPTER 2

The Beginnings of Mathematics in Greece

Thales was the first to go to Egypt and bring back to Greece this study [geometry]; he himself discovered many propositions, and disclosed the underlying principles of many others to his successors, in some cases his method being more general, in others more empirical.

—Proclus's *Summary* (c. 450 CE) of Eudemus's *History* (c. 320 BCE)[1]

A report from a visit to Egypt with Plato by Simmias of Thebes in 379 BCE (from a dramatization by Plutarch of Chaeronea (first–second century CE)): "On our return from Egypt a party of Delians met us . . . and requested Plato, as a geometer, to solve a problem set them by the god in a strange oracle. The oracle was to this effect: the present troubles of the Delians and the rest of the Greeks would be at an end when they had doubled the altar at Delos. As they not only were unable to penetrate its meaning, but failed absurdly in constructing the altar . . . , they called on Plato for help in their difficulty. Plato . . . replied that the god was ridiculing the Greeks for their neglect of education, deriding, as it were, our ignorance and bidding us engage in no perfunctory study of geometry; for no ordinary or near-sighted intelligence, but one well versed in the subject, was required to find two mean proportionals, that being the only way in which a body cubical in shape can be doubled with a similar increment in all dimensions. This would be done for them by Eudoxus of Cnidus . . . ; they were not, however, to suppose that it was this the god desired, but rather that he was ordering the entire Greek nation to give up war and its miseries and cultivate the Muses, and by calming their passions through the practice of discussion and study of mathematics, so to live with one another that their relationships should be not injurious, but profitable."[2]

From Chapter 2 of *A History of Mathematics*, Third Edition. Victor J. Katz.

As the quotation and the (probably) fictional account indicate, a new attitude toward mathematics appeared in Greece sometime before the fourth century BCE. It was no longer sufficient merely to calculate numerical answers to problems. One now had to prove that the results were correct. To double a cube, that is, to find a new cube whose volume was twice that of the original one, is equivalent to determining the cube root of 2, and that was not a difficult problem numerically. The oracle, however, was not concerned with numerical calculation, but with geometric construction. That in turn depended on geometric proof by some logical argument, the earliest manifestation of such in Greece being attributed to Thales.

This change in the nature of mathematics, beginning around 600 BCE, was related to the great differences between the emerging Greek civilization and those of Egypt and Babylonia, from whom the Greeks learned. The physical nature of Greece with its many mountains and islands is such that large-scale agriculture was not possible. Perhaps because of this, Greece did not develop a central government. The basic political organization was the *polis*, or city-state. The governments of the city-state were of every possible variety but in general controlled populations of only a few thousand. Whether the governments were democratic or monarchical, they were not arbitrary. Each government was ruled by law and therefore encouraged its citizens to be able to argue and debate. It was perhaps out of this characteristic that there developed the necessity for proof in mathematics, that is, for argument aimed at convincing others of a particular truth.

Because virtually every city-state had access to the sea, there was constant trade, both in Greece itself and with other civilizations. As a result, the Greeks were exposed to many different peoples and, in fact, themselves settled in areas all around the eastern Mediterranean. In addition, a rising standard of living helped to attract able people from other parts of the world. Hence, the Greeks were able to study differing answers to fundamental questions about the world. They began to create their own answers. In many areas of thought, they learned not to accept what had been handed down from ancient times. Instead, they began to ask, and to try to answer, "Why?" Greek thinkers eventually came to the realization that the world around them was knowable, that they could discover its characteristics by rational inquiry. Hence, they were anxious to discover and expound theories in such fields as mathematics, physics, biology, medicine, and politics. And although Western civilization owes a great debt to Greek society in literature, art, and architecture, it is to Greek mathematics that we owe the idea of mathematical proof, an idea at the basis of modern mathematics and, by extension, at the foundation of our modern technological civilization.

This chapter discusses the Greek numerical system and then considers the contributions of the earliest Greek mathematicians beginning in the sixth century BCE. It then deals with the beginnings of the Greek approach to geometric problem solving and concludes with the work of Plato and Aristotle in the fourth century BCE on the nature of mathematics and the idea of logical reasoning.

2.1 THE EARLIEST GREEK MATHEMATICS

Unlike the situation with Egyptian and Babylonian mathematics, there are virtually no extant texts of Greek mathematics that were actually written in the first millennium BCE. What we have today are copies of copies of copies, where the actual written documents date from not much earlier than 1000 CE. And even then, the earliest complete texts (of which these are copies) are not from earlier than about 300 BCE. So to tell the story of early Greek mathematics, we are forced to rely on works that were originally written much later than the actual occurrences. Thus, given that these works do not always agree with each other, there is a considerable amount of controversy about some of the early developments. We will try to present the story as coherently as possible, but will note many areas in which scholarly opinion varies.

2.1.1 Greek Numbers

From what fragments exist from ancient times, and even from some of the copies, we do know that the Greeks represented numbers in a ciphered system using their alphabet, from as far back as the sixth century BCE. The representation was as shown in Table 2.1, where the letters ς (digamma) for 6, ϙ (koppa) for 90, and ϡ (sampi) for 900 are letters that by this time were no longer in use. Hence, 754 was written $\psi\nu\delta$ and 293 was written σϙγ. To represent thousands, a mark was made to the left of the letters α through θ; for example, $'\theta$ represented 9000. Larger numbers still were written using the letter M to represent myriads (10,000), with the number of myriads written above: $M^{\delta} = 40{,}000$, $M^{'\zeta\rho o\epsilon} = 71{,}750{,}000$.

TABLE 2.1 ***Representation of a number system used by the Greeks as early as the sixth century BCE.***

Letter	Value	Letter	Value	Letter	Value
α	1	ι	10	ρ	100
β	2	κ	20	σ	200
γ	3	λ	30	τ	300
δ	4	μ	40	υ	400
ϵ	5	ν	50	ϕ	500
ς	6	ξ	60	χ	600
ζ	7	o	70	ψ	700
η	8	π	80	ω	800
θ	9	ϙ	90	ϡ	900

Among the earliest extant inscriptions in this alphabetic cipher were numbers inscribed on the walls of the tunnel on the island of Samos constructed by Eupalinus around 550 BCE to bring water from a spring outside the capital city through a mountain to a point inside the city walls. Modern archaeological excavations of the tunnel have revealed that it was dug by two teams that met in the middle (Fig. 2.1). There are no records as to how the construction crews managed to keep digging in the correct direction, but there have been many theories as to how this was done. The latest archaeological evidence leads to the conclusion that the builders used the simplest possible mathematical techniques, such as lining up flags to make sure that the diggers kept digging in the right direction. And evidently the numbers on the walls, 10, 20, 30, . . . , 200 (from the south entrance) and 10, 20, 30, . . . , 300 (from the north entrance) were written to keep tabs on the distances dug. Although most of the tunnel is

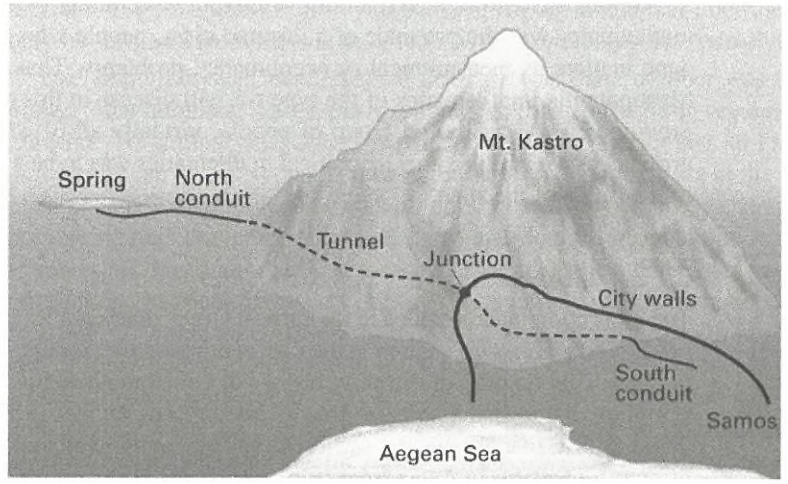


FIGURE 2.1
Water tunnel on the island of Samos

straight, there is one clear jog in the tunnel, probably necessitated by difficult soil conditions. Somehow, Eupalinus managed to figure out at that point how to get the digging back to the correct direction.

The numbers in the Eupalinus tunnel are integers. But Greek merchants and accountants, for example, needed fractions as well. Generally, in this early period, the Greeks used the Egyptian system of "parts." There was a special symbol $\angle$, which represented a half; $\acute{\beta}$ represented two-thirds. For the rest, the system was standard: $\acute{\gamma}$ represented one-third, $\acute{\delta}$ one-fourth, and so on. More complicated fractions than simple parts are expressed as the sum of an integer and different simple parts. For example, the fraction we represent as 12/17 might be represented as $\angle\acute{\iota}\acute{\beta}\acute{\iota}\acute{\zeta}\acute{\lambda}\acute{\delta}\acute{\upsilon}\acute{\alpha}\acute{\xi}\acute{\eta}$, which in modern notation would be $\frac{1}{2}+\frac{1}{12}+\frac{1}{17}+\frac{1}{34}+\frac{1}{51}+\frac{1}{68}$. We do not know if there was any systematic method for figuring out which unit fractions to use, for there are many possible ways to represent 12/17, or as the Greeks would say, the "seventeenth part of twelve." In addition, there is clearly the possibility of confusion between the representations of, for example, $\frac{1}{20}+\frac{1}{5}$ and $\frac{1}{25}$. But all those who needed to calculate evidently had methods of determining how they would use this system and how to avoid confusion.[3]

Fortunately for us, most of the early Greek mathematics we will discuss involves little calculation. As Aristotle wrote in his *Metaphysics*,

> At first, he who invented any art whatever that went beyond the common perceptions of man was naturally admired by men, not only because there was something useful in the inventions, but because he was thought wise and superior to the rest. But as more arts were invented, and some were directed to the necessities of life, others to recreation, the inventors of the latter were naturally always regarded as wiser than the inventors of the former, because their branches of knowledge did not aim at utility. Hence when all such inventions were already established, the sciences which do not aim at giving pleasure or at the necessities of life were discovered, and first in the places where men first began to have leisure. This is why the mathematical arts were founded in Egypt; for there the priestly caste was allowed to be at leisure.[4]

Although Aristotle referred only to Egypt, he certainly believed that in Greece as well mathematics was the province of a leisured class, people who did not deal with such mundane matters as measurement or accountancy problems. Thus, in Greece as in Egypt and Mesopotamia, mathematics of the type we will discuss in this chapter and the next was the province of a very limited group of people, virtually all of whom were part of the ruling groups. As we will see, this theoretical mathematics was to be a central part of the education of the rulers of the state.

FIGURE 2.2
Thales on a Greek stamp

2.1.2 Thales

The most complete reference to the earliest Greek mathematics is in the commentary to Book I of Euclid's *Elements* written in the fifth century CE by Proclus, some 800 to 1000 years after the fact. This account of the early history of Greek mathematics is generally thought to be a summary of a formal history written by Eudemus of Rhodes in about 320 BCE, the original of which is lost. In any case, the earliest Greek mathematician mentioned is Thales (c. 624–547 BCE), from Miletus in Asia Minor (Fig. 2.2). There are many stories recorded about him, most written down several hundred years after his death. These include his prediction of a solar eclipse in 585 BCE and his application of the angle-side-angle criterion of triangle congruence to the problem of measuring the distance to a ship at sea. He is said to have impressed Egyptian officials by determining the height of a pyramid by comparing the length of its shadow to that of the length of the shadow of a stick of known height. Thales is also credited with discovering the theorems that the base angles of an isosceles triangle are equal and that vertical angles are equal and with proving that the diameter of a circle divides the circle into two equal parts. Although exactly how Thales "proved" any of these results is not known, it does seem clear that he advanced some logical arguments.

Aristotle related the story that Thales was once reproved for wasting his time on idle pursuits. Therefore, noticing from certain signs that a bumper crop of olives was likely in a particular year, he quietly cornered the market on oil presses. When the large crop in fact was harvested, the olive growers all had to come to him for presses. He thus demonstrated that a philosopher or a mathematician could in fact make money if he thought it worthwhile. Whether this or any of the other stories are literally true is not known. In any case, the Greeks of the fourth century BCE and later credited Thales with beginning the Greek mathematical tradition. In fact, he is generally credited with beginning the entire Greek scientific enterprise, including recognizing that material phenomena are governed by discoverable laws.

FIGURE 2.3
Pythagoras on a Greek coin

2.1.3 Pythagoras and His School

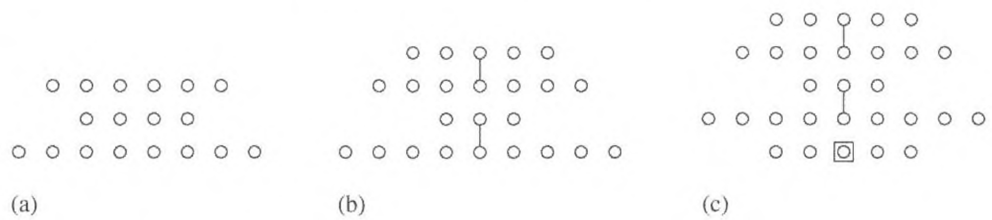
There are also extensive but unreliable stories about Pythagoras (c. 572–497 BCE), including that he spent much time not only in Egypt, where Thales was said to have visited, but also in Babylonia (Fig. 2.3). Around 530 BCE, after having been forced to leave his native Samos, he settled in Crotona, a Greek town in southern Italy. There he gathered around him a group of disciples, later known as the Pythagoreans, in what was considered both a religious order and a philosophical school. From the surviving biographies, all written centuries after his death, we can infer that Pythagoras was probably more of a mystic than a rational thinker, but one who commanded great respect from his followers. Since there are no extant works

ascribed to Pythagoras or the Pythagoreans, the mathematical doctrines of his school can only be surmised from the works of later writers, including the "neo-Pythagoreans."

One important such mathematical doctrine was that "number was the substance of all things," that numbers, that is, positive integers, formed the basic organizing principle of the universe. What the Pythagoreans meant by this was not only that all known objects have a number, or can be ordered and counted, but also that numbers are at the basis of all physical phenomena. For example, a constellation in the heavens could be characterized by both the number of stars that compose it and its geometrical form, which itself could be thought of as represented by a number. The motions of the planets could be expressed in terms of ratios of numbers. Musical harmonies depend on numerical ratios: two plucked strings with ratio of length 2 : 1 give an octave, with ratio 3 : 2 give a fifth, and with ratio 4 : 3 give a fourth. Out of these intervals an entire musical scale can be created. Finally, the fact that triangles whose sides are in the ratio of 3 : 4 : 5 are right-angled established a connection of number with angle. Given the Pythagoreans' interest in number as a fundamental principle of the cosmos, it is only natural that they studied the properties of positive integers, what we would call the elements of the theory of numbers.

The starting point of this theory was the dichotomy between the odd and the even. The Pythagoreans probably represented numbers by dots or, more concretely, by pebbles. Hence, an even number would be represented by a row of pebbles that could be divided into two equal parts. An odd number could not be so divided because there would always be a single pebble left over. It was easy enough using pebbles to verify some simple theorems. For example, the sum of any collection of even numbers is even, while the sum of an even collection of odd numbers is even and that of an odd collection is odd (Fig. 2.4).

FIGURE 2.4
(a) The sum of even numbers is even. (b) An even sum of odd numbers is even. (c) An odd sum of odd numbers is odd.



Among other simple corollaries of the basic results above were the theorems that the square of an even number is even, while the square of an odd number is odd. Squares themselves could also be represented using dots, providing simple examples of "figurate" numbers. If one represents a given square in this way, for example, the square of 4, it is easy to see that the next higher square can be formed by adding a row of dots around two sides of the original figure. There are $2 \cdot 4 + 1 = 9$ of these additional dots. The Pythagoreans generalized this observation to show that one can form squares by adding the successive odd numbers to 1. For example, $1 + 3 = 2^2$, $1 + 3 + 5 = 3^2$, and $1 + 3 + 5 + 7 = 4^2$. The added odd numbers were in the L shape generally called a gnomon (Fig. 2.5). Other examples of figurate numbers include the triangular numbers, also shown in Figure 2.5, produced by successive additions of the natural numbers themselves. Similarly, oblong numbers, numbers of the form $n(n + 1)$, are produced by beginning with 2 and adding the successive even numbers (Fig. 2.6). The first

FIGURE 2.5
Square and triangular numbers

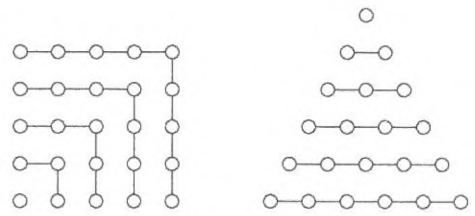


FIGURE 2.6
Oblong numbers

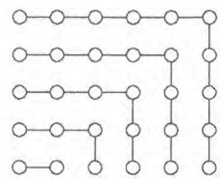
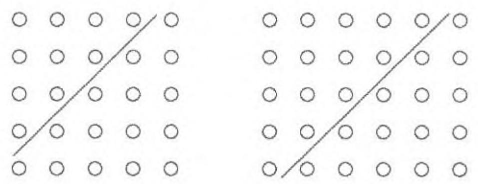


FIGURE 2.7
Two theorems on triangular numbers



four of these are 2, 6, 12, and 20, that is, 1×2, 2×3, 3×4, and 4×5. Figure 2.7 provides easy demonstrations of the results that any oblong number is the double of a triangular number and that any square number is the sum of two consecutive triangular numbers.

Another number theoretical problem of particular interest to the Pythagoreans was the construction of Pythagorean triples. There is evidence that they saw that for an odd number n, the triple $(n, \frac{n^2-1}{2}, \frac{n^2+1}{2})$ is a Pythagorean triple, while if m is even, $(m, (\frac{m}{2})^2 - 1, (\frac{m}{2})^2 + 1)$ is such a triple. An explanation of how the Pythagoreans may have demonstrated the first of these results from their dot configurations begins with the remark that any odd number is the difference of two consecutive squares. Hence, if the odd number is itself a square, then three square numbers have been found such that the sum of two equals the third (Fig. 2.8). To find the sides of these squares, the Pythagorean triple itself, note that the side of the gnomon is given since it is the square of an odd number. The side of the smaller square is found by subtracting 1 from the gnomon and halving the remainder. The side of the larger square is one more than that of the smaller. A similar proof can be given for the second result. Although there is no explicit testimony to additional results involving Pythagorean triples, it seems probable that the Pythagoreans considered the odd and even properties of these triples. For example, it is not difficult to prove that in a Pythagorean triple, if one of the terms is odd, then two of them must be odd and one even.

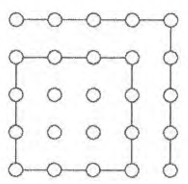


FIGURE 2.8
An odd square that is the difference of two squares

The geometric theorem out of which the study of Pythagorean triples grew, namely, that in any right triangle the square on the hypotenuse is equal to the sum of the squares on the legs, has long been attributed to Pythagoras himself, but there is no direct evidence of this. The theorem was known in other cultures long before Pythagoras lived. Nevertheless, it was

the knowledge of this theorem by the fifth century BCE that led to the first discovery of what is today called an irrational number.

For the early Greeks, number always was connected with things counted. Because counting requires that the individual units must remain the same, the units themselves can never be divided or joined to other units. In particular, throughout formal Greek mathematics, a number meant a "multitude composed of units," that is, a counting number. Furthermore, since the unit 1 was not a multitude composed of units, it was not considered a number in the same sense as the other positive integers. Even Aristotle noted that two was the smallest "number."

Because the Pythagoreans considered number as the basis of the universe, everything could be counted, including lengths. In order to count a length, of course, one needed a measure. The Pythagoreans thus assumed that one could always find an appropriate measure. Once such a measure was found in a particular problem, it became the unit and thus could not be divided. In particular, the Pythagoreans assumed that one could find a measure by which both the side and diagonal of a square could be counted. In other words, there should exist a length such that the side and diagonal were integral multiples of it. Unfortunately, this turned out not to be true. The side and diagonal of a square are **incommensurable**; there is no common measure. Whatever unit of measure is chosen such that an exact number will fit the length of one of these lines, the other line will require some number plus a portion of the unit, and one cannot divide the unit. (In modern terms, this result is equivalent to the statement that the square root of two is irrational.) We do not know who discovered this result, but scholars believe that the discovery took place in approximately 430 BCE. And although it is frequently stated that this discovery precipitated a crisis in Greek mathematics, the only reliable evidence shows that the discovery simply opened up the possibility of some new mathematical theories. In fact, Aristotle wrote in his *Metaphysics*,

> For all men begin, as we said, by wondering that things are as they are, as they do about self-moving marionettes, or about the solstices or the incommensurability of the diagonal of a square with the side; for it seems wonderful to all who have not yet seen the reason, that there is a thing which cannot be measured even by the smallest unit. But we must end in the contrary and, according to the proverb, the better state, as is the case in these instances too when men learn the cause; for there is nothing which would surprise a geometer so much as if the diagonal turned out to be commensurable.[5]

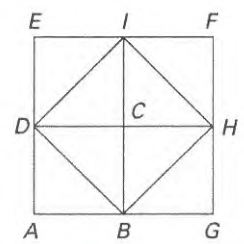


FIGURE 2.9
The incommensurability of the side and idagonal of a square (first possibility)

In other words, Aristotle seems to say that although the incommensurability is initially surprising, once one finds the reason—and clearly Greek thinkers did so—it then becomes very unsurprising.

So what is the "cause" of the incommensurability and how did a Greek thinker discover it? The only hint is in another work of Aristotle, who notes that if the side and diagonal are assumed commensurable, then one may deduce that odd numbers equal even numbers. One possibility as to the form of the discovery is the following: Assume that the side BD and diagonal DH in Figure 2.9 are commensurable, that is, that each is represented by the number of times it is measured by their common measure. It may be assumed that at least one of these numbers is odd, for otherwise there would be a larger common measure. Then the squares $DBHI$ and $AGFE$ on the side and diagonal, respectively, represent square numbers. The latter square is clearly double the former, so it represents an even square number. Therefore, its side $AG = DH$ also represents an even number and the square $AGFE$ is a multiple of four. Since $DBHI$ is half of $AGFE$, it must be a multiple of two; that is, it represents an even square. Hence, its side BD must also be even. But this contradicts the original assumption, that one of DH, BD, must be odd. Therefore, the two lines are incommensurable.

It must be realized that such a proof presupposes that by this time the notion of proof was ingrained into the Greek conception of mathematics. Although there is no evidence that the Greeks of the fifth century BCE possessed the entire mechanism of an axiomatic system and had explicitly recognized that certain statements need to be accepted without proof, they certainly had decided that some form of logical argument was necessary for determining the truth of a particular result. Furthermore, this entire notion of incommensurability represents a break from the Babylonian and Egyptian concepts of calculation with numbers. There is naturally no question that one can assign a numerical value to the length of the diagonal of a square of side one unit, as the Babylonians did, but the notion that no "exact" value can be found is first formally recognized in Greek mathematics.

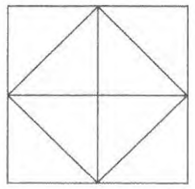


FIGURE 2.10

Dissection argument for determining the diagonal of a square

Although the Greeks could not "measure" the diagonal of a square, that line, as a geometric object, was still significant. Plato, in his dialogue *Meno*, had Socrates question a slave boy about finding a square whose area is double that of square of side two feet. The boy first suggests that each side should be doubled. Socrates pointed out that this would give a square of area sixteen. The boy's second guess, that the new side should be three feet, is also evidently incorrect. So Socrates then led him to figure out that if one draws a diagonal of the original square and then constructs a square on that diagonal, the new square is exactly double the old one. But Socrates' proof of this is simply by a dissection argument (Fig. 2.10). There is no mention of the length of this diagonal at all.[6]

2.1.4 Squaring the Circle and Doubling the Cube

The idea of proof and the change from numerical calculation are further exemplified in the mid-fifth century attempts to solve two geometric problems, problems that were to occupy Greek mathematicians for centuries: the squaring of the circle (already attempted in Egypt) and the duplication of the cube (as noted in the oracle). The multitude of attacks on these particular problems and the slightly later one of trisecting an arbitrary angle serve to remind us that a central goal of Greek mathematics was geometrical problem solving, and that, to a large extent, the great body of theorems found in the major extant works of Greek mathematics served as logical underpinnings for these solutions. Interestingly, that these problems apparently could not be solved via the original tools of straightedge and compass was known to enough of the Greek public that Aristophanes could refer to "squaring the circle" as something absurd in his play *The Birds*, first performed in 414 BCE.

Hippocrates of Chios (mid-fifth century BCE) (no connection to the famous physician) was among the first to attack the cube and circle problems. As to the first of these, Hippocrates perhaps realized that the problem was analogous to the simpler problem of doubling a square of side a. That problem could be solved by constructing a mean proportional b between a and $2a$, a length b such that $a : b = b : 2a$, for then $b^2 = 2a^2$. From the fragmentary records of Hippocrates' work, it is evident that he was familiar with performing such constructions. In any case, ancient accounts record that Hippocrates was the first to come up with the idea of reducing the problem of doubling the cube of side a to the problem of finding two mean proportionals b, c, between a and $2a$. For if $a : b = b : c = c : 2a$, then

$$a^3 : b^3 = (a : b)^3 = (a : b)(b : c)(c : 2a) = a : 2a = 1 : 2$$

and $b^3 = 2a^3$. Hippocrates was not, however, able to construct the two mean proportionals using the geometric tools at his disposal. It was left to some of his successors to find this construction.

Hippocrates similarly made progress in the squaring of the circle, essentially by showing that certain lunes (figures bounded by arcs of two circles) could be "squared," that is, that their areas could be shown equal to certain regions bounded by straight lines. To do this, he first had to show that the areas of circles are to one another as the squares on their diameters, a fact evidently known to the Babylonian scribes. How he accomplished this is not known. In any case, he could now square the lune on a quadrant of a circle.

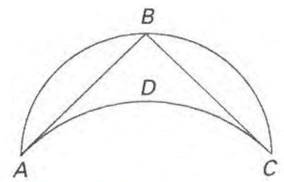


FIGURE 2.11

Hippocrates' lune on an isosceles right triangle

Suppose that semicircle ABC is circumscribed about the isosceles right triangle ABC and that around the base AC an arc ADC of a circle is drawn so that segment ADC is similar to segments AB and BC; that is, the arcs of each are the same fraction of a circle, in this case, one-quarter (Fig. 2.11). It follows from the result on areas of circles that similar segments are also to one another as the squares on their chords. Therefore, segment ADC is equal to the sum of segments AB and BC. If we add to each of these areas the part of the triangle outside arc ADC, it follows that the lune $ABCD$ is equal to the triangle ABC.

Although Hippocrates gave constructions for squaring other lunes or combinations of lunes, he was unable to actually square a circle. Nevertheless, it is apparent that his attempts on the squaring problem and the doubling problem were based on a large collection of geometric theorems, theorems that he organized into the first recorded book on the elements of geometry.

2.2 THE TIME OF PLATO

The time of Plato (429–347 BCE) (Fig. 2.12) saw significant efforts made toward solving the problems of doubling the cube and squaring the circle and toward dealing with incommensurability and its impact on the theory of proportion. These advances were achieved partly because Plato's Academy, founded in Athens around 385 BCE, drew together scholars from all over the Greek world. These scholars conducted seminars in mathematics and philosophy with small groups of advanced students and also conducted research in mathematics, among other fields. There is an unverifiable story, dating from some 700 years after the school's founding, that over the entrance to the Academy was inscribed the Greek phrase ΑΓΕΩΜΕΤΡΗΤΟΣ ΜΗΔΕΙΣ ΕΙΣΙΤΩ, meaning roughly, "Let no one ignorant of geometry enter here." A student "ignorant of geometry" would also be ignorant of logic and hence unable to understand philosophy.

FIGURE 2.12

Plato and Aristotle: A detail of Raphael's painting *The School of Athens*

The mathematical syllabus inaugurated by Plato for students at the Academy is described by him in his most famous work, *The Republic*, in which he discussed the education that should be received by the philosopher-kings, the ideal rulers of a state. The mathematical part of this education was to consist of five subjects: arithmetic (that is, the theory of numbers), plane geometry, solid geometry, astronomy, and harmonics (music). The leaders of the state are "to practice calculation, not like merchants or shopkeepers for purposes of buying and selling, but with a view to war and to help in the conversion of the soul itself from the world of becoming to truth and reality. . . . It will further our intentions if it is pursued for the sake of knowledge and not for commercial ends. . . . It has a great power of leading the mind upwards and forcing it to reason about pure numbers, refusing to discuss collections of material things which can be seen and touched."[7] In other words, arithmetic is to be studied for the training of the mind (and incidentally for its military usefulness). The arithmetic of which Plato writes includes not only the Pythagorean number theory already discussed but also additional material that is included in Books VII–IX of Euclid's *Elements* and will be considered later.

Again, a limited amount of plane geometry is necessary for practical purposes, particularly in war, when a general must be able to lay out a camp or extend army lines. But even though mathematicians talk of operations in plane geometry such as squaring or adding, the object of geometry, according to Plato, is not to *do* something but to gain knowledge, "knowledge, moreover, of what eternally exists, not of anything that comes to be this or that at some time and ceases to be."[8] So, as in arithmetic, the study of geometry—and for Plato this means theoretical, not practical, geometry—is for "drawing the soul towards truth." It is importanat to mention here that Plato distinguished carefully between, for example, the real geometric circles drawn by people and the essential or ideal circle, held in the mind, which is the true object of geometric study. In practice, one cannot draw a circle and its tangent with only one point in common, although this is the nature of the mathematical circle and the mathematical tangent.

The next subject of mathematical study should be solid geometry. Plato complained in the *Republic* that this subject has not been sufficiently investigated. This is because "no state thinks [it] worth encouraging" and because "students are not likely to make discoveries without a director, who is hard to find."[9] Nevertheless, Plato felt that new discoveries would be made in this field, and, in fact, much was done between the dramatic date of the dialogue (about 400 BCE) and the time of Euclid, some of which is included in Books XI–XIII of the *Elements*.

In any case, a decent knowledge of solid geometry was necessary for the next study, that of astronomy, or, as Plato puts it, "solid bodies in circular motion." Again, in this field Plato distinguished between the stars as material objects with motions showing accidental irregularities and variations and the ideal abstract relations of their paths and velocities expressed in numbers and perfect figures such as the circle. It is this mathematical study of ideal bodies that is the true aim of astronomical study. Thus, this study should take place by means of problems and without attempting to actually follow every movement in the heavens.

Similarly, a distinction is made in the final subject, of harmonics, between material sounds and their abstraction. The Pythagoreans had discovered the harmonies that occur when strings are plucked together with lengths in the ratios of certain small positive integers. But in encouraging his philosopher-kings in the study of harmonics, Plato meant for them to go beyond the actual musical study, using real strings and real sounds, to the abstract level of "inquiring which numbers are inherently consonant and which are not, and for what reasons."[10] That is, they should study the mathematics of harmony, just as they should study the mathematics of astronomy, and should not be overly concerned with real stringed instruments or real stars. It turns out that a principal part of the mathematics necessary in both studies is the theory of ratio and proportion, the subject matter of Euclid's *Elements*, Book V.

Although it is not known whether the entire syllabus discussed by Plato was in fact taught at the Academy, it is certain that Plato brought in the best mathematicians of his day to teach and do research, including Theaetetus (c. 417–369 BCE) and Eudoxus (c. 408–355 BCE), who

we will discuss later. The most famous person associated with the Academy, however, was Aristotle.

2.3 ARISTOTLE

Aristotle (384–322 BCE) (Fig. 2.13) studied at Plato's Academy in Athens from the time he was 18 until Plato's death in 347. Shortly thereafter, he was invited to the court of Philip II of Macedon to undertake the education of Philip's son Alexander, who soon after his own accession to the throne in 335 began his successful conquest of the Mediterranean world (Fig. 2.14). Meanwhile, Aristotle returned to Athens where he founded his own school, the Lyceum, and spent the rest of his days writing, lecturing, and holding discussions with his advanced students. Although Aristotle wrote on many subjects, including politics, ethics, epistemology, physics, and biology, his strongest influence as far as mathematics was concerned was in the area of logic.

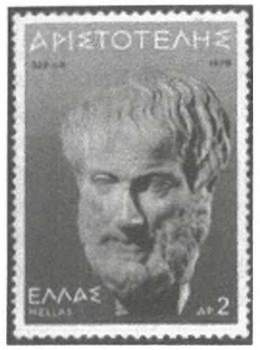


FIGURE 2.13
Bust of Aristotle

FIGURE 2.14
Painting of Alexander on horseback

2.3.1 Logic

Although there is only fragmentary evidence of logical argument in mathematical works before the time of Euclid, some appearing in the work of Hippocrates already mentioned, it is apparent that from at least the sixth century BCE, the Greeks were developing the notions of logical reasoning. The active political life of the city-states encouraged the development of argumentation and techniques of persuasion. And there are many examples from philosophical works, especially those of Parmenides (late sixth century BCE) and his disciple Zeno of Elea (fifth century BCE), that demonstrate various detailed techniques of argument. In particular, there are examples of such techniques as *reductio ad absurdum*, in which one assumes that a proposition to be proved is false and then derives a contradiction, and *modus tollens*, in which one shows first that if A is true, then B follows, shows next that B is not true, and concludes finally that A is not true. It was Aristotle, however, who took the ideas developed over the centuries and first codified the principles of logical argument.

Aristotle believed that logical arguments should be built out of **syllogisms**, where "a syllogism is discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so."[11] In other words, a syllogism consists of certain statements that are taken as true and certain other statements that are then necessarily true. For example, the argument "if all monkeys are primates, and all primates are mammals, then it follows that all monkeys are mammals," exemplifies one type of syllogism, whereas the argument "if all Catholics are Christians and no Christians are Moslem, then it follows that no Catholic is Moslem," exemplifies a second type.

After clarifying the principles of dealing with syllogisms, Aristotle noted that syllogistic reasoning enables one to use "old knowledge" to impart new. If one accepts the premises of a syllogism as true, then one must also accept the conclusion. One cannot, however, obtain every piece of knowledge as the conclusion of a syllogism. One has to begin somewhere with truths that are accepted without argument. Aristotle distinguished between the basic truths that are peculiar to each particular science and the ones that are common to all. The former are often called **postulates**, while the latter are known as **axioms**. As an example of a common truth, he gave the axiom "take equals from equals and equals remain." His examples of peculiar truths for geometry are "the definitions of line and straight." By these he presumably meant that one postulates the existence of straight lines. Only for the most basic ideas did Aristotle permit the postulation of the object defined. In general, however, whenever one defines an object, one must in fact prove its existence. "For example, arithmetic assumes the meaning of odd and even, square and cube, geometry that of incommensurable, . . . , whereas the existence of these attributes is demonstrated by means of the axioms and from previous conclusions as premises."[12] Aristotle also listed certain basic principles of argument, principles that earlier thinkers had used intuitively. One such principle is that a given assertion cannot be both true and false. A second principle is that an assertion must be either true or false; there is no other possibility.

For Aristotle, logical argument according to his methods is the only certain way of attaining scientific knowledge. There may be other ways of gaining knowledge, but demonstration via a series of syllogisms is the one way by which one can be sure of the results. Because one cannot prove everything, however, one must always be careful that the premises, or axioms, are true and well known As Aristotle wrote, "syllogism there may indeed be without these conditions, but such syllogism, not being productive of scientific knowledge, will not be demonstration."[13] In other words, one can choose any axioms one wants and draw conclusions from them, but if one wants to attain knowledge, one must start with "true" axioms. The question then becomes, how can one be sure that one's axioms are true? Aristotle answered that these primary premises are learned by induction, by drawing conclusions from our own sense perception of numerous examples. This question of the "truth" of the basic axioms has been discussed by mathematicians and philosophers ever since Aristotle's time. On the other hand, Aristotle's rules of attaining knowledge by beginning with axioms and using demonstrations to gain new results has become the model for mathematicians to the present day.

Although Aristotle emphasized the use of syllogisms as the building blocks of logical arguments, Greek mathematicians apparently never used them. They used other forms, as have most mathematicians down to the present. Why Aristotle therefore insisted on syllogisms is not clear. The basic forms of argument actually used in mathematical proof were analyzed in some detail in the third century BCE by the Stoics, of whom the most prominent was Chrysippus (280–206 BCE). This form of logic is based on **propositions**, statements that can be either true or false, rather than on the Aristotelian syllogisms. The basic rules of inference dealt with by Chrysippus, with their traditional names, are the following, where p, q, and r stand for propositions:

(1) *Modus ponens*
If p, then q.
p.
Therefore, q.

(2) *Modus tollens*
If p, then q.
Not q.
Therefore, not p.

(3) *Hypothetical syllogism*
If p, then q.
If q, then r.
Therefore, if p, then r.

(4) *Alternative syllogism*
p or q.
Not p.
Therefore, q.

For example, from the statements "if it is daytime, then it is light" and "it is daytime," one can conclude by *modus ponens* that "it is light." From "if it is daytime, then it is light" and "it is not light," one concludes by *modus tollens* that "it is not daytime." Adding to the first hypothesis the statement "if it is light, then I can see well," one concludes by the hypothetical syllogism that "if it is daytime, then I can see well." Finally, from "either it is daytime or it is nighttime" and "it is not daytime," the rule of the alternative syllogism allows us to conclude that "it is nighttime."

2.3.2 Number versus Magnitude

Another of Aristotle's contributions was the introduction into mathematics of the distinction between number and magnitude. The Pythagoreans had insisted that all was number, but Aristotle rejected that idea. Although he placed number and magnitude in a single category, "quantity," he divided this category into two classes, the discrete (number) and the continuous (magnitude). As examples of the latter, he cited lines, surfaces, bodies, and time. The primary distinction between these two classes is that a magnitude is "that which is divisible into divisibles that are infinitely divisible,"[14] while the basis of number is the indivisible unit. Thus, magnitudes cannot be composed of indivisible elements, whereas numbers inevitably are.

Aristotle further clarified this idea in his definition of "in succession" and "continuous." Things are **in succession** if there is nothing of their own kind intermediate between them. For example, the numbers 3 and 4 are in succession. Things are **continuous** when they touch and when "the touching limits of each become one and the same."[15] Line segments are therefore continuous if they share an endpoint. Points cannot make up a line, because they would have to be in contact and share a limit. Since points have no parts, this is impossible. It is also impossible for points on a line to be in succession, that is, for there to be a "next point." For between two points on a line is a line segment, and one can always find a point on that segment.

Today, a line segment is considered to be composed of an infinite collection of points, but to Aristotle this would make no sense. He did not conceive of a completed or actual infinity. Although he used the term "infinity," he only considered it as potential. For example, one can bisect a continuous magnitude as often as one wishes, and one can count these bisections. But in neither case does one ever come to an end. Furthermore, mathematicians really do not need infinite quantities such as infinite straight lines. They only need to postulate the existence of, for example, arbitrarily long straight lines.

2.3.3 Zeno's Paradoxes

One of the reasons Aristotle had such an extended discussion of the notions of infinity, indivisibles, continuity, and discreteness was that he wanted to refute the famous paradoxes of Zeno. Zeno stated these paradoxes, perhaps in an attempt to show that the then current notions of motion were not sufficiently clear, but also to show that any way of dividing space or time must lead to problems. The first paradox, the *Dichotomy*, "asserts the non-existence of motion on the ground that that which is in locomotion must arrive at the half-way stage before it arrives at the goal."[16] (Of course, it must then cover the half of the half before it reaches the middle, etc.) The basic contention here is that an object cannot cover a finite distance by moving during an infinite sequence of time intervals. The second paradox, the *Achilles*, asserts a similar point: "In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead."[17] Aristotle, in refuting the paradoxes, concedes that time, like distance, is infinitely divisible. But he is not bothered by an object covering an infinity of intervals in a finite amount of time. For "while a thing in a finite time cannot come in contact with things quantitatively infinite, it can come in contact with things infinite in respect to divisibility, for in this sense time itself is also infinite."[18] In fact, given the motion in either of these paradoxes, one can calculate when one will reach the goal or when the fastest runner will overtake the slowest.

Zeno's third and fourth paradoxes show what happens when one asserts that a continuous magnitude is composed of indivisible elements. The *Arrow* states that "if everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless."[19] In other words, if there are such things as indivisible instants, the arrow cannot move during that instant. Since if, in addition, time is composed of nothing but instants, then the moving arrow is always at rest. Aristotle refutes this paradox by noting that not only are there no such things as indivisible instants, but motion itself can only be defined in a period of time. A modern refutation, on the other hand, would deny the first premise because motion is now defined by a limit argument.

The paradox of the *Stadium* supposes that there are three sets of identical objects: the A's at rest, the B's moving to the right past the A's, and the C's moving to the left with equal velocity. Suppose the B's have moved one place to the right and the C's one place to the left, so that B_1, which was originally under A_4, is now under A_5, while C_1, originally under A_5, is now under A_4 (Fig. 2.15). Zeno supposes that the objects are indivisible elements of space and that they move to their new positions in an indivisible unit of time. But since there must have been a moment at which B_1 was directly over C_1, there are two possibilities. Either the two objects did not cross, and so there was no motion at all, or in the indivisible instant, each object had occupied two separate positions, so that the instant was in fact not indivisible. Aristotle believed that he had refuted this paradox because he had already denied the original assumption—that time is composed of indivisible instants.

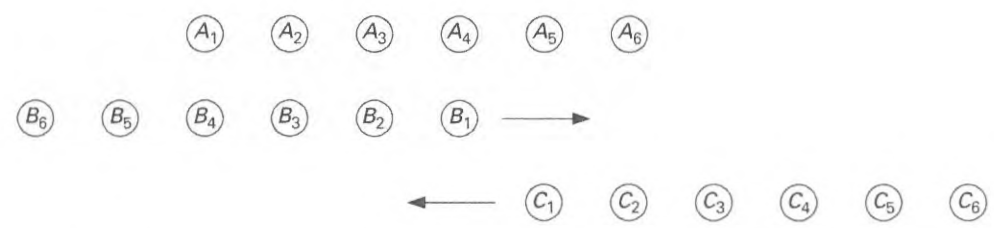


FIGURE 2.15

Zeno's parardox of the *Stadium*

Interestingly, the four paradoxes exhaust the four possibilities of divisibility/indivisibility of space and time. That is, in the *Arrow* both space and time are assumed infinitely divisible, in the *Stadium* both are assumed ultimately indivisible, in the *Dichotomy* space is assumed divisible and time indivisible, and in the *Achilles* the reverse is assumed. So Zeno has shown each of the four possibilities leads to a contradiction.

Controversy regarding these paradoxes has lasted throughout history. The ideas contained in Zeno's statements and Aristotle's attempts at refutation have been extremely fruitful in

forcing mathematicians to the present day to think carefully about their assumptions in dealing with the concepts of the infinite or the infinitely small. And in Greek times they were probably a significant factor in the development of the distinction between continuous magnitude and discrete number so important to Aristotle and ultimately to Euclid.

EXERCISES

1. Represent 125, 62, 4821, and 23,855 in the Greek alphabetic notation.

2. Represent 8/9 as a sum of distinct unit fractions. Express the result in the Greek notation. Note that the answer to this problem is not unique.

3. Represent 200/9 as the sum of an integer and distinct unit fractions. Express the result in Greek notation.

4. There are extant Greek land surveys that give measurements of fields and then find the area so the land can be assessed for tax purposes. In general, areas of quadrilateral fields were approximated by multiplying together the averages of the two pairs of opposite sides. In one document, one pair of sides is given as $a = 1/4 + 1/8 + 1/16 + 1/32$ and $c = 1/8 + 1/16$, where the lengths are in fractions of a *schonion*, a measure of approximately 150 feet. The second pair of sides is given as $b = 1/2 + 1/4 + 1/8$ and $d = 1$. Find the average of a and c, the average of b and d, and multiply them together to show that the area of the field is approximately $1/4 + 1/16$ square *schonion*. Note that the taxman has rounded up the exact answer (presumably to collect more taxes).

5. Thales is said to have invented a method of finding distances of ships from shore by use of the angle-side-angle theorem. Here is a possible method: Suppose A is a point on shore and S is a ship (Fig. 2.16). Measure the distance AC along a perpendicular to AC and bisect it at B. Draw CE at right angles to AC and pick point E on it in a straight line with B and S. Show that $\triangle EBC \cong \triangle SBA$ and therefore that $SA = EC$.

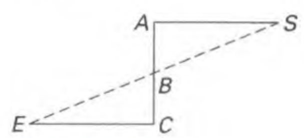


FIGURE 2.16

One method Thales could have used to determine the distance to a ship at sea

6. A second possibility for Thales' method is the following: Suppose Thales was atop a tower on the shore with an instrument made of a straight stick and a crosspiece AC that could be rotated to any desired angle and then would remain where it was put (Fig. 2.17). One rotates AC until one sights the ship S, then turns and sights an object T on shore without moving the crosspiece. Show that $\triangle AET \cong \triangle AES$ and therefore that $SE = ET$.

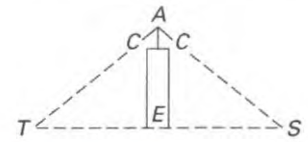


FIGURE 2.17

Second method Thales could have used to determine the distance to a ship at sea

7. Suppose Thales found that at the time a stick of length 6 feet cast a shadow of 9 feet, there was a length of 342 feet from the edge of the pyramid's side to the tip of its shadow. Suppose further that the length of a side of the pyramid was 756 feet. Find the height of the pyramid. (Assuming that the pyramid is laid out so the sides are due north-south and due east-west, this method requires that the sun be exactly in the south when the measurement is taken. When does this occur?[20])

8. Show that the nth triangular number is represented algebraically as $T_n = \frac{n(n+1)}{2}$ and therefore that an oblong number is double a triangular number.

9. Show algebraically that any square number is the sum of two consecutive triangular numbers.

10. Show using dots that eight times any triangular number plus 1 makes a square. Conversely, show that any odd square diminished by 1 becomes eight times a triangular number. Show these results algebraically as well.

11. Show that in a Pythagorean triple, if one of the terms is odd, then two of them must be odd and one even.

CHAPTER 3

Euclid

Not much younger than these [Hermotimus of Colophon and Philippus of Mende, students of Plato] is Euclid, who put together the Elements*, collecting many of Eudoxus's theorems, perfecting many of Theaetetus's, and also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors. This man lived in the time of the first Ptolemy.*

—Proclus's *Summary* (c. 450 CE) of Eudemus's *History* (c. 320 BCE)[1]

Two legends about Euclid: Ptolemy is said to have asked him if there was any shorter way to geometry than through the *Elements*, and he replied that there was "no royal road to geometry." And, according to Stobaeus (fifth century CE), a student, after learning the first theorem, asked Euclid, "What shall I get by learning these things?" Euclid then asked his slave to give the student a coin, "since he must make gain out of what he learns."[2]

FIGURE 3.1

Euclid (detail from Raphael's painting *The School of Athens*). Note that there is no evidence of Euclid's actual appearance.

Since the first Ptolemy, Ptolemy I Soter, the Macedonian general of Alexander the Great who became ruler of Egypt after the death of Alexander in 323 BCE and lived until 283 BCE, it is generally assumed from the quotation from Proclus that Euclid flourished around 300 BCE (Fig. 3.1). But besides this date, written down some 750 years later, there is nothing at all known about the life of the author of the *Elements*. Nevertheless, most historians believe that Euclid was one of the first scholars active at the Museum and Library at Alexandria, founded by Ptolemy I and his successor, Ptolemy II Philadelphus. "Museum" here means a "Temple of the Muses," that is, a location where scholars meet and discuss philosophical and literary ideas. The Museum was to be, in effect, a government research establishment. The Fellows of the Museum received stipends and free board and were exempt from taxation. In this way the rulers of Egypt hoped that men of eminence would be attracted there from the entire Greek world. In fact, the Museum and Library soon became a focal point of the highest developments in Greek scholarship, both in the humanities and the sciences. The Fellows were initially appointed to carry on research, but since younger students gathered there as well, the Fellows soon turned to teaching. The aim of the Library was to collect the entire body of Greek literature in the best available copies and to organize it systematically. Ship captains who sailed from Alexandria were instructed to bring back scrolls from every port they touched until their return. The story is told that Ptolemy III, who reigned from 247–221 BCE, borrowed the authorized texts of the playwrights Aeschylus, Sophocles, and Euripides from Athens against a large deposit. But rather than return the originals, he returned only copies. He was quite willing to forfeit the deposit. The Library ultimately contained over 500,000 volumes in every field of knowledge. Although parts of the library were destroyed in various wars, some of it remained intact until the fourth century CE.

This chapter will be devoted primarily to a study of Euclid's most important work, the *Elements*, but we will also consider Euclid's *Data*.

3.1 INTRODUCTION TO THE *ELEMENTS*

The *Elements* of Euclid is the most important mathematical text of Greek times and probably of all time. It has appeared in more editions than any work other than the *Bible*. It has been translated into countless languages and has been continuously in print in one country or another nearly since the beginning of printing. Yet to the modern reader the work is incredibly dull. There are no examples; there is no motivation; there are no witty remarks; there is no calculation. There are simply definitions, axioms, theorems, and proofs. Nevertheless, the book has been intensively studied. Biographies of many famous mathematicians indicate that Euclid's work provided their initial introduction into mathematics, that it in fact excited them and motivated them to become mathematicians. It provided them with a model of how "pure mathematics" should be written, with well-thought-out axioms, precise definitions, carefully stated theorems, and logically coherent proofs. Although there were earlier versions of *Elements* before that of Euclid, his is the only one to survive, perhaps because it was the first one written after both the foundations of proportion theory and the theory of irrationals had been developed and the careful distinctions always to be made between number and magnitude had been propounded by Aristotle. It was therefore both "complete" and well organized. Since the mathematical community as a whole was of limited size, once Euclid's work was recognized for its general excellence, there was no reason to keep another inferior work in circulation.

Euclid wrote his text about 2300 years ago. There are, however, no copies of the work dating from that time. The earliest extant fragments include some potsherds discovered in Egypt dating from about 225 BCE, on which are written what appear to be notes on two propositions from Book XIII, and pieces of papyrus containing parts of Book II dating from about 100 BCE. Copies of the work were, however, made regularly from the time of Euclid. Various editors made emendations, added comments, or put in new lemmas. In particular, Theon of Alexandria (fourth century CE) was responsible for one important new edition. Most of the extant manuscripts of Euclid's *Elements* are copies of this edition. The earliest such copy now in existence is in the Bodleian Library at Oxford University and dates from 888. There is, however, one manuscript in the Vatican Library, dating from the tenth century, which is not a copy of Theon's edition but of an earlier version. It was from a detailed comparison of this manuscript with several old manuscript copies of Theon's version that the Danish scholar J. L. Heiberg compiled a definitive Greek version in the 1880s, as close to what he believed the Greek original was as possible. The extracts to be discussed here are all adapted from Thomas Heath's 1908 English translation of Heiberg's Greek. (It should be noted that some modern scholars believe that one can get closer to Euclid's original by taking more account of medieval Arab translations than Heiberg was able to do.)

Euclid's *Elements* is a work in thirteen books. The first six books form a relatively complete treatment of two-dimensional geometric magnitudes while Books VII–IX deal with the theory of numbers, in keeping with Aristotle's instructions to separate the study of magnitude and number. In fact, Euclid included two entirely separate treatments of proportion theory—in Book V for magnitudes and in Book VII for numbers. Book X then provides the link between the two concepts, because it is here that Euclid introduced the notions of commensurability and incommensurability and showed that, with regard to proportions, commensurable magnitudes may be treated as if they were numbers. The book continues by presenting a classification of some incommensurable magnitudes. Euclid next dealt in Book XI with three-dimensional geometric objects and in Book XII with the method of exhaustion applied both to two- and three-dimensional objects. Finally, in Book XIII he constructed the five regular polyhedra and classified some of the lines involved according to his scheme of Book X.

It is useful to note that much of the ancient mathematics discussed in Chapter 1 is included in one form or another in Euclid's masterwork, with the exception of actual methods of arithmetic computation. The methodology, however, is entirely different. Namely, mathematics in earlier cultures always involves numbers and measurement. Numerical algorithms for solving various problems are prominent. The mathematics of Euclid, however, is completely nonarithmetical. There are no numbers used in the entire work aside from a few small positive integers. There is also no measurement. Various geometrical objects are compared, but not by use of numerical measures. There are no cubits or acres or degrees. The only measurement standard—for angles—is the right angle. Nevertheless, the question must be asked as to how much influence the mathematical cultures of Egypt and Mesopotamia had on Euclidean mathematics. In this chapter we discuss certain pieces of evidence in this regard, but a complete answer to this question cannot yet be given.

SIDEBAR 3.1 *Euclid's Postulates and Common Notions*

Postulates

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to one another.
5. That, if a straight line intersecting two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

Common Notions (Axioms)

1. Things which are equal to the same thing are also equal to one another.
2. If equals are added to equals, the wholes are equal.
3. If equals are subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

3.2 BOOK I AND THE PYTHAGOREAN THEOREM

As Aristotle suggested, a scientific work needs to begin with definitions and axioms. Euclid therefore prefaced several of the thirteen books with definitions of the mathematical objects discussed, most of which are relatively standard. He also prefaced Book I with ten axioms; five of them are geometrical postulates and five are more general truths about mathematics called "common notions." Euclid then proceeded to prove one result after another, each one based on the previous results and/or the axioms. If one reads Book I from the beginning, one never has any idea what will come next. It is only when one gets to the end of the book, where Euclid proved the Pythagorean Theorem, that one realizes that Book I's basic purpose is to lead to the proof of that result. Thus, in order to understand the reasons for various theorems, we begin our discussion of Book I with the Pythagorean Theorem and work backwards. This also enables us to see why certain unproved results must be assumed, namely, the axioms. Sidebar 3.1 does, however, list all of Euclid's axioms (called "postulates" and "common notions") and Sidebar 3.2 has selected definitions.

As we discuss the various propositions, the reader should keep in mind a few important issues. First, although Euclid has modeled the overall structure of the *Elements* using some of Aristotle's ideas, he did not use syllogisms in his proofs. His proofs were written out in natural language and generally used the notions of propositional logic. In fact, one can find examples of all four of the basic rules of inference among Euclid's proofs. Next, Euclid always assumed that if he proved a result for a particular configuration representing the hypotheses of the theorem and illustrated in a diagram, he had proved the result generally. For example, as we will see, he proved the Pythagorean Theorem by drawing some lines and marking some points on a particular right triangle, then arguing to his result on that triangle, and then concluding that the result is true for any right triangle. Of course, when mathematicians today use that strategy, they base it on explicit ideas of mathematical logic. Euclid, in contrast, never discussed his philosophy of proof; he just went ahead and proved

CHAPTER 4

Archimedes and Apollonius

The third book [of Conics] contains many incredible theorems of use for the construction of solid loci and for limits of possibility of which the greatest part and the most beautiful are new. And when we had grasped these, we knew that the three-line and four-line locus had not been constructed by Euclid, but only a chance part of it and that not very happily. For it was not possible for this construction to be completed without the additional things found by us.

—Preface to Book I of Apollonius's *Conics*[1]

Here is a story told by Vitruvius: "It is no surprise that Hiero [the king of Syracuse in the third century BCE], after he had obtained immense kingly power in Syracuse, decided, because of the favorable turn of events, to dedicate a votive crown of gold to the immortal gods in a certain shrine. He contracted for the craftsman's wages, and he [himself] weighed out the gold precisely for the contractor. This contractor completed the work with great skill and on schedule; it was approved by the king, and the contractor seemed to have used up the furnished supply of gold. Later, charges were leveled that in the making of the crown a certain amount of gold had been removed and replaced by an equal amount of silver. Hiero, outraged that he should have been shown so little respect, and not knowing by what method he might expose the theft, requested that Archimedes take the matter under consideration on his behalf. Now Archimedes, once he had charge of this matter, chanced to go to the baths, and there, as he stepped into the tub, he noticed that however much he immersed his body in it, that much water spilled over the sides of the tub. When the reason for this occurence came clear to him, he did not hesitate, but in a transport of joy he leapt out of the tub, and as he rushed home naked, he let one and all know that he had truly found what he had been looking for—because as he ran he shouted over and over in Greek: 'I found it! I found it! [Eureka! Eureka!]' "[2]

From Chapter 4 of *A History of Mathematics*, Third Edition. Victor J. Katz.

Greek mathematics in the third and early second centuries BCE was dominated by two major figures, Archimedes of Syracuse (c. 287–212 BCE) and Apollonius of Perga (c. 250–175 BCE), each heir to a different aspect of fourth-century Greek mathematics. The former took over the "limit" methods of Eudoxus and succeeded not only in applying them to determine areas and volumes of new figures, but also in developing new techniques that enabled the results to be discovered in the first place. Archimedes, unlike his predecessors, was neither reluctant to share his methods of discovery nor afraid of performing numerical calculations and exhibiting numerical results. And also, unlike Euclid, he did not write systematic treatises on a major subject, but instead what may be considered research monographs, treatises concentrating on the solution of a particular set of problems. These treatises were often sent originally as letters to mathematicians Archimedes knew, so many of them include prefaces describing the circumstances and purposes of their writing. Furthermore, several of the treatises presented mathematical models of certain aspects of what we would call theoretical physics and applied his physical principles to the invention of various mechanical devices.

Apollonius, on the other hand, was instrumental in extending the domain of analysis to new and more difficult geometric construction problems. As a foundation for these new approaches, he created his magnum opus, the *Conics*, a work in eight books developing synthetically the important properties of this class of curves, properties that were central in developing new solutions to such problems as the duplication of the cube and the trisection of the angle.

As is the case for Euclid, there are no surviving manuscripts of the works of either Archimedes or Apollonius dating from anywhere near their time of composition. For Archimedes, we know that an edition of some of his works with extensive commentaries was prepared by Eutocius early in the sixth century somewhere near Byzantium. This edition was the basis for some part of the three collections of Archimedes' works, written on parchment, that were available in Byzantium in the tenth or eleventh century. Only one of these is still extant and will be discussed in some detail below. The second oldest extant Archimedes manuscript is a 1260 Latin translation by Moerbeke, probably made from both of the two now missing Byzantine copies, but such a literal translation that from it we can practically re-create the Greek text. There are also several fifteenth- and sixteenth-century Greek copies of the missing Byzantine versions. Heiberg collated these manuscripts in the late nineteenth century and produced the now standard Greek text of Archimedes in 1880–81, with a revised version in 1910–15. Similarly, Eutocius prepared an edition of the first four books of Apollonius's *Conics* of which the Greek manuscripts available in tenth-century Byzantium were copies. The earliest surviving Greek manuscript was copied there in the twelfth or thirteenth century. But there are two older Arabic manuscripts of seven books of the *Conics*, one written in Egypt in the early eleventh century and now in Istanbul, and one written in Maragha toward the end of that century and now in Oxford. Again, Heiberg produced a definitive Greek edition of Books I–IV in 1891–93, while a definitive Arabic edition of Books V–VII was only produced in 1990 by Toomer.

This chapter surveys the extant works of both of these mathematicians, as well as the work of certain others who considered similar problems.

4.1 ARCHIMEDES AND PHYSICS

Archimedes was the first mathematician to derive quantitative results from the creation of mathematical models of physical problems on earth. In particular, Archimedes is responsible for the first proof of the law of the lever (Fig. 4.1) and its application to finding centers of gravity, as well as the first proof of the basic principle of hydrostatics and some of its important applications.

FIGURE 4.1
Archimedes and the law of the lever

4.1.1 The Law of the Lever

Everyone is familiar with the principle of the lever from having played on seesaws as children. Equal weights at equal distances from the fulcrum of the lever balance, and a lighter child can balance a heavier one by being farther away. The ancients were aware of this principle as well. The law even appears in writing in a work on mechanics attributed to Aristotle: "Since the greater radius is moved more quickly than the less by an equal weight, and there are three elements in the lever, the fulcrum . . . and two weights, that which moves and that which is moved, therefore the ratio of the weight moved to the moving weight is the inverse ratio of their distances from the fulcrum."[3]

As far as is known, no one before Archimedes had created a mathematical model of the lever by which one could derive a mathematical proof of the law of the lever. In general, a difficulty in attempting to apply mathematics to physical problems is that the physical situation is often quite complicated. Therefore, the situation needs to be idealized. One ignores those aspects that appear less important and concentrates on only the essential variables of the physical problem. This idealization is referred to today as the creation of a mathematical model. The lever is a case in point. To deal with it as it actually occurs, one would need to consider not only the weights applied to the two ends and their distances from the fulcrum, but also the weight and composition of the lever itself. It may be heavier at one end than the other. Its thickness may vary. It may bend slightly—or even break—when certain weights are applied at certain points. In addition, the fulcrum is also a physical object of a certain size. The lever may slip somewhat along the fulcrum, so it may not be clear from what point the distance of the weights should be measured. To include all of these factors in a mathematical analysis of the lever would make the mathematics extremely difficult. Archimedes therefore simplified the physical situation. He assumed that the lever itself was rigid, but weightless, and that the fulcrum and the weights were mathematical points. He was then able to develop the mathematical principles of the lever.

Archimedes dealt with these principles at the beginning of his treatise *Planes in Equilibrium*. Being well trained in Greek geometry, he began by stating seven postulates he would assume, four of which are reproduced here.

1. Equal weights at equal distances are in equilibrium, and equal weights at unequal distances are not in equilibrium but incline toward the weight that is at the greater distance.
2. If, when weights at certain distances are in equilibrium, something is added to one of the weights, they are not in equilibrium but incline toward the weight to which the addition was made.

BIOGRAPHY

Archimedes (287–212 BCE)

More biographical information about Archimedes survives than about any other Greek mathematician. Much is found in Plutarch's biography of the Roman general Marcellus, who captured Syracuse, the major city of Sicily, after a siege in 212 BCE during the Second Punic War. Other Greek and Roman historians also discuss aspects of Archimedes' life.

Archimedes was the son of the astronomer Phidias and perhaps a relative of King Hiero II of Syracuse, under whose rule from 270 to 216 BCE the city greatly flourished. It is also probable that Archimedes spent time in his youth in Alexandria, for he is credited with the invention there of what is known as the Archimedean screw, a machine for raising water used for irrigation (Fig. 4.2). Moreover, the prefaces of many of his works are addressed to scholars at Alexandria, including one of the chief librarians, Eratosthenes. Most of his life, however, was spent in his native Syracuse, where he was repeatedly called upon to use his mathematical talents to solve various practical problems for Hiero and his successor. Many stories are recorded about his intense dedication to his work. Plutarch, in *The Lives of the Noble Grecians and Romans* (*Great Books*, 14, Dryden translation), wrote that on many occasions his concentration on mathematics "made him forget his food and neglect his person, to that degree that when he was carried by absolute violence to bathe or have his body anointed, he used to trace geometrical figures in the ashes of the fire, and diagrams in the oil on his body, being in a state of entire preoccupation, and in the truest sense, divine possession with his love and delight in science" (p. 254). And it was this dedication that ultimately cost him his life.

His genius as a military engineer kept the Roman army under Marcellus at bay for months during the siege of Syracuse. Finally, however, probably through treachery, the Romans were able to enter the city. Marcellus gave explicit orders that Archimedes be spared, but Plutarch relates that, "as fate would have it, he was intent on working out some problem with a diagram and, having fixed his mind and his eyes alike on his investigation, he never noticed the incursion of the Romans nor the capture of the city. And when a soldier came up to him suddenly and bade him follow to Marcellus, he refused to do so until he had worked out his problem to a demonstration; whereat the soldier was so enraged that he drew his sword and slew him" (*Lives*, p. 252).

FIGURE 4.2
Archimedes and the Archimedean screw

3. Similarly, if anything is taken away from one of the weights, they are not in equilibrium but incline toward the weight from which nothing was taken.
6. If magnitudes at certain distances are in equilibrium, other magnitudes equal to them will also be in equilibrium at the same distances.

These postulates come from basic experience with levers. The first postulate, in fact, is an example of what is usually called the **Principle of Insufficient Reason**. That is, one assumes that equal weights at equal distances balance because there is no reason to make any other assumption. The lever cannot incline to the right, for example, since what is the right side from one viewpoint is the left side from another. The second and third postulates are equally obvious. The sixth appears to be virtually meaningless. In Archimedes' use of it, however, it appears that the second clause means "other equal magnitudes, the centers of gravity of which lie at the same distances from the fulcrum, will also be in equilibrium." That is, the influence of a magnitude on the lever depends solely on its weight and the position of its center of gravity.

Although Archimedes used the term "center of gravity" in many of the book's propositions, he never gave a definition. Presumably, he felt that the concept was so well known to his readers that a definition was unnecessary. There are, however, later Greek texts that do give a definition, perhaps the one that was even used in Archimedes' time: "We say that the center of gravity of any body is a point within that body which is such that, if the body be conceived to be suspended from that point, the weight carried thereby remains at rest and preserves the original position."[4] But it was also clear to Archimedes, and this is what he expressed in postulate 6, that the downward tendency of gravitation may be thought of as being concentrated in that one point. Note that in neither the postulates nor the theorems is there any mention of the lever itself. It is just there. Its weight does not enter into the calculations. Archimedes in effect assumed that the lever is weightless and rigid. Its only motion is inclination to one side or the other.

The first two in Archimedes' sequence of propositions leading to the law of the lever are very easy:

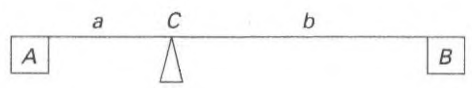
PROPOSITION 1 *Weights which balance at equal distances are equal.*

PROPOSITION 2 *Unequal weights at equal distances will not balance but will incline toward the greater weight.*

The proof of the first result is by *reductio ad absurdum*. For if the weights are not equal, take away from the greater the difference between the two. By postulate 3, the remainders will not balance. This contradicts postulate 1, since now we have equal weights at equal distances. Our original assumption must then be false. To prove Proposition 2, again take away from the greater weight the difference between the two. By postulate 1, the remainders will balance. So if this difference is added back, the lever will incline toward the greater by postulate 2.

PROPOSITION 3 *Suppose A and B are unequal weights with $A > B$ which balance at point C (Fig. 4.3). Let $AC = a$, $BC = b$. Then $a < b$. Conversely, if the weights balance and $a < b$, then $A > B$.*

FIGURE 4.3
Planes in Equilibrium, Proposition 3



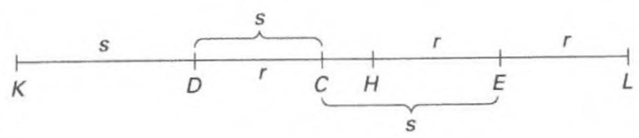
The proof is again by contradiction. Suppose $a \not< b$. Subtract from A the difference $A - B$. By postulate 3, the lever will incline toward B. But if $a = b$, the equal remainders will balance, and if $a > b$, the lever will incline toward A by postulate 1. These two contradictions imply that $a < b$. The proof of the converse is equally simple.

In Propositions 4 and 5, Archimedes showed that the center of gravity of a system of two (and three) equally spaced equal weights is at the geometric center of the system. These results are extended in the corollaries to any system of equally spaced weights provided that those at equal distance from the center are equal. The law of the lever itself is stated in Propositions 6 and 7:

PROPOSITION 6, 7 *Two magnitudes, whether commensurable [Proposition 6] or incommensurable [Proposition 7], balance at distances inversely proportional to the magnitudes.*

First assume that the magnitudes A, B, are commensurable; that is, $A : B = r : s$, where r, s, are numbers. Archimedes' claim is that if A is placed at E and B at D, and if C is taken on DE with $DC : CE = r : s$, then C is the center of gravity of the two magnitudes A, B (Fig. 4.4). To prove the result, assume that units have been chosen so that $DC = r$ and $CE = s$. Choose H on DE so that $HE = r$ and extend the line past E to L so that EL also equals r. Also extend the line in the opposite direction to K, making $DK = HD = s$. Then C is the midpoint of LK. Now break A into $2r$ equal parts and B into $2s$ equal parts. Space the first set equally along LH and the second along HK. Since $A : B = r : s = 2r : 2s$, it follows that each part of A is equal to each part of B. From the corollary mentioned above, the center of gravity of the parts of A will be at the midpoint E of HL, while the center of gravity of the parts of B will be at the midpoint D of KH. By postulate 6, nothing is changed if A itself is considered situated at E and B at D. On the other hand, the total system consists of $2r + 2s$ equal parts equally spaced along the line KL. Hence, the center of gravity of the system is at the midpoint C of that line. Therefore, weight A placed at E and weight B placed at D balance about the point C.

FIGURE 4.4
Planes in Equilibrium, Proposition 6



Archimedes concluded the proof in the incommensurable case by a *reductio* argument using the fact that if two magnitudes are incommensurable, one can subtract from the first an amount smaller than any given quantity such that the remainder is commensurable with the second. Interestingly enough, Archimedes made no use here of the Eudoxian proportion theory for incommensurables of *Elements*, Book V, nor even of Theaetetus's earlier version based on the Euclidean algorithm. He instead made use essentially of a continuity argument. But even so, his proof is somewhat flawed.

Nevertheless, Archimedes used the law of the lever in the remainder of the treatise to find the centers of gravity of various geometrical figures. He proved that the center of gravity of a parallelogram is at the intersection of its diagonals, of a triangle at the intersection of two medians, and of a parabolic segment at a point on the diameter three-fifths of the distance from the vertex to the base.

4.1.2 Applications to Engineering

Not only are there geometric consequences of the law of the lever, but there are also physical consequences. In particular, given any two weights A and B and any lever, there is always a point C at which the weights balance. If A is much heavier than B, they will balance when A is sufficiently close to C and B is sufficiently far away. But then any additional weight added to B will incline the lever in that direction and will cause weight A to be lifted. Archimedes therefore was able to boast that "any weight might be moved and . . . if there were another earth, by going into it he could move this one."[5] When King Hiero heard of this boast, he asked Archimedes to demonstrate his principles in actual experiment. Archimedes complied, but instead of using a lever, he probably made use of some kind of pulley or tackle system,

which also provided a great mechanical advantage. Plutarch wrote that "he fixed accordingly upon a ship of burden out of the king's arsenal, which could not be drawn out of the dock without great labor and many men; and loading her with many passengers and a full freight, sitting himself the while far off, with no great endeavor, but only holding the head of the pulley in his hand and drawing the cords by degrees, he drew the ship in a straight line, as smoothly and evenly as if she had been in the sea."[6] Other sources give a variant of Plutarch's story, to the effect that Archimedes was responsible for the construction of a magnificent ship, named the *Syracusa*, and singlehandedly launched this 4200-ton luxury vessel.

Archimedes enjoyed the greatest fame in antiquity, however, for his design of various engines of war. These engines enabled Syracuse to hold off the Roman siege for many months. Archimedes devised various missile launchers as well as huge cranes by which he was able to lift Roman ships out of the water and dash them against the rocks or simply dump out the crew. In fact, he was so successful that any time the Romans saw a little rope or piece of wood come out from the walls of the city, they fled in panic.

Plutarch related that Archimedes was not particularly happy as an engineer: "He would not deign to leave behind him any commentary or writing on such subjects; but, repudiating as sordid and ignoble the whole trade of engineering, and every sort of art that lends itself to mere use and profit, he placed his whole affection and ambition in those purer speculations where there can be no reference to the vulgar needs of life."[7] In fact, however, there is evidence that Archimedes did write on certain mechanical subjects, including a book *On Sphere Making* in which he described his planetarium, a mechanical model of the motions of the heavenly bodies, and another one on water clocks.

The incident of the gold crown and the bath led Archimedes to the study of an entirely new subject, that of hydrostatics, in which he discovered its basic law, that a solid heavier than a fluid will, when weighed in the fluid, be lighter than its true weight by the weight of the fluid displaced. It is, however, not entirely clear how Archimedes' noticing the water being displaced in his bath led him to the concept of weight being lessened. Perhaps he also noticed that his body felt lighter in the water.

As in his study of levers, Archimedes began the mathematical development of hydrostatics, in his treatise *On Floating Bodies*, by giving a simplifying postulate. He was then able to show, among other results, that the surface of any fluid at rest is the surface of a sphere whose center is the same as that of the earth. He could then deal with solids floating or sinking in fluids by assuming that the fluid was part of a sphere. Archimedes was able to solve the crown problem by using the basic law, proved as Proposition 7. One way by which he could have applied the law is suggested by Heath, based on a description in a Latin poem of the fifth century CE.[8] Suppose the crown is of weight W, composed of unknown weights w_1 and w_2 of gold and silver, respectively. To determine the ratio of gold to silver in the crown, first weigh it in water and let F be the loss of weight. This amount can be determined by weighing the water displaced. Next take a weight W of pure gold and let F_1 be its weight loss in water. It follows that the weight of water displaced by a weight w_1 of gold is $\frac{w_1}{W}F_1$. Similarly, if the weight of water displaced by a weight W of pure silver is F_2, the weight of water displaced by a weight w_2 of silver is $\frac{w_2}{W}F_2$. Therefore, $\frac{w_1}{W}F_1 + \frac{w_2}{W}F_2 = F$. Thus, the ratio of gold to silver is given by

$$\frac{w_1}{w_2} = \frac{F - F_2}{F_1 - F}.$$

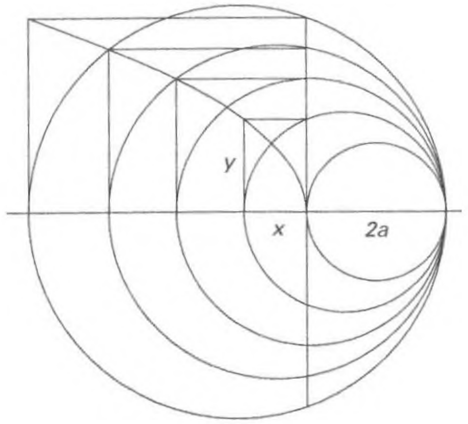


FIGURE 4.15
Euclidean pointwise construction of a parabola

connected, the desired curve is drawn.[16] We note that although each point of this curve has been constructed using Euclidean tools, the completed curve is not a proper construction in Euclid's sense. In any case, it does appear that the conic sections were introduced as tools for the solution of certain geometric problems.

There can be only speculation as to how the Greeks realized that curves useful in solving the cube doubling problem could be generated as sections of a cone. Someone, perhaps Menaechmus himself, may have noticed that the circle diagram above could be thought of as a diagram of level curves of a certain cone, hence that the curve could be generated by a section of such a cone. Another possibility is that these curves appeared as the path of the moving shadow of the gnomon on a sundial as the sun traveled through its circular daily path, which in turn was one base of a double cone whose vertex was the tip of the gnomon. In this suggestion, the plane in which the shadow falls would be the cutting plane. It might further have been noted that the apparent shape of a circle viewed from a point outside its plane was an ellipse, and this shape comes from a plane cutting the cone of vision. In any case, by the end of the fourth century, there were in existence two extensive treatises on the properties of the curves obtained as sections of cones, one by Aristaeus (fourth century BCE) and one by Euclid. Although neither is still available, a good deal about their contents can be inferred from Archimedes' extensive references to basic theorems on conic sections.

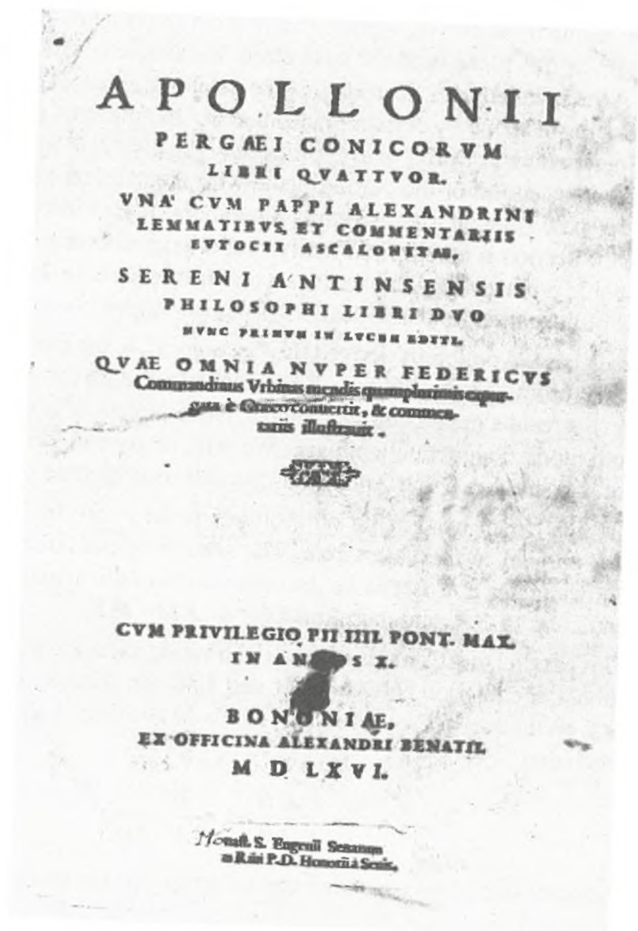
Recall that Euclid (in Book XI of the *Elements*) defined a cone as a solid generated by rotating a right triangle about one of its legs. He then classified the cones in terms of their vertex angles as right angled, acute angled, or obtuse angled. A section of such a cone can be formed by cutting the cone by a plane at right angles to the generating line, the hypotenuse of the given right triangle. The "section of a right-angled cone" is today called a parabola, the "section of an acute-angled cone" an ellipse, and the "section of an obtuse-angled cone" a hyperbola. The names in quotation marks are those generally used by Archimedes and his predecessors.

BIOGRAPHY

Apollonius (250–175 BCE)

Apollonius was born in Perga, a town in southern Asia Minor, but few details are known about his life. Most of the reliable information comes from the prefaces to the various books of his magnum opus, the *Conics* (Fig. 4.16). These indicate that he went to Alexandria as a youth to study with successors of Euclid and probably remained there for most of his life, studying, teaching, and writing. He became famous in ancient times first for his work on astronomy, but later for his mathematical work, most of which is known today only by titles and summaries in works of later authors. Fortunately, seven of the eight books of the *Conics* do survive, and these represent in some sense the culmination of Greek mathematics. It is difficult for us today to comprehend how Apollonius could discover and prove the hundreds of beautiful and difficult theorems without modern algebraic symbolism. Nevertheless, he did so, and there is no record of any later Greek mathematical work that approaches the complexity or intricacy of the *Conics*.

FIGURE 4.16
Title page of the first Latin printed edition of Apollonius's *Conics*, 1566 (*Source*: Smithsonian Institution Libraries, Photo No. 86-4346)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VNA CVM PAPPI ALEXANDRINI
LEMMATIBVS, ET COMMENTARIIS
EVTOCII ASCALONITÆ.
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