

Niels Henrik Abel, *Untersuchungen über die Reihe:*

$$1 + \frac{m}{1}x + \frac{m \cdot (m-1)}{1 \cdot 2} \cdot x^2 + \frac{m \cdot (m-1) \cdot (m-2)}{1 \cdot 2 \cdot 3} \cdot x^3 + \dots \quad 1643$$


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I.

[3]

If one investigates the reasoning that is usually applied when infinite series are dealt with, then he will find that in general it is not very satisfactory, and that the number of theorems about infinite series that can be regarded as rigorously established is very modest. As a rule one applies the methods of the analysis to the infinite series exactly as if they were finite. I do not find this permissible without a particular proof. If for instance two series are to be multiplied with each other, then one posits

$$\begin{aligned} &(u_0 + u_1 + u_2 + u_3 + \dots)(v_0 + v_1 + v_2 + v_3 + \dots) = \\ &u_0 v_0 + (u_0 v_1 + u_1 v_0) + (u_0 v_2 + u_1 v_1 + u_2 v_0) + \dots \\ &+ (u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + \dots + u_n v_0) + \dots \end{aligned}$$

This equation is fully correct, if both series

$$u_0 + u_1 + \dots \quad \text{and} \quad v_0 + v_1 + \dots$$

are finite. However, if they are infinite, then, firstly, they must by necessity *converge*,<sup>1644</sup> because a divergent series has no sum, and then the series on the right-hand side of the above equation must also *converge*. Only with these restrictions is the above equation correct. If I am not mistaken, this condition has so far not been taken into account. This is going to happen in the present study. In the same way an abundance [4] of similar operations will need to be justified, for instance the usual procedure by which a magnitude is divided by an infinite series, an infinite series is raised to a power, its logarithm, its sine, its cosine are taken, etc.

[...]

II.

First we shall establish some necessary theorems regarding series. Here we shall be guided by *Cauchy's* excellent work "*Cours d'analyse de l'école polytechnique*", which should be read by any analyst who loves rigour in mathematical demonstrations.

*Explanation.* An arbitrary series

$$v_0 + v_1 + v_2 + \dots + v_m + \dots$$

will be called convergent if, for ever-increasing values of  $m$ , the sum  $v_0 + v_1 + v_2 + \dots + v_m$  approaches a given limit with any precision asked for. This limit will be called the *sum of the series*. In the opposite case the series will be called divergent, then it has no sum. From this definition follows that, if the series is to converge, the sum  $v_m + v_{m+1} + \dots + v_{m+n}$  approaches zero with any precision asked for [still for ever-increasing values of  $m/n$ ], irrespective of the value of  $n$ .